

Lecture 1: April 3

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Ising Model

Comes from statistical mechanics, and was originally applied in 1924 to study ferromagnets. Let $\Lambda \subseteq \mathbb{Z}^d$. Let $\sigma_i \in \{-1, 1\}$ denote the "spin" at vertex $i \in \Lambda$. Define a probability measure μ_Λ on the space of configurations $\Sigma \equiv \{-1, 1\}^\Lambda$ with density proportional to $\exp(\beta \sum_{i,j \in \eta_\Lambda} \sigma_i \sigma_j)$ where η_Λ is the set of all pairs of neighboring vertices in Λ . Setting $\beta = 0$ yields the i.i.d measure while setting $\beta > 0$ tries to get neighboring spins aligned in the same direction.

Phase Transitions of the Ising Model

Expectations of the Ising Model

$$\mathbf{E}_{\mu_\Lambda}(\sigma_i \sigma_j) \equiv Z^{-1} \sum_{\sigma \in \{-1, 1\}^\Lambda} \sigma_i \sigma_j \exp(\beta H(\sigma))$$

$$H(\sigma) \equiv \sum_{i,j \in \eta_\Lambda} \sigma_i \sigma_j$$

$$Z \equiv \sum_{\sigma \in \{-1, 1\}^\Lambda} \exp(\beta H(\sigma))$$

Ising proved that there is never any magnetization when $d = 1$. More precisely, $|\mathbf{E}_{\mu_\Lambda}(\sigma_i \sigma_j)| \leq c \exp(-c(\beta)|i - j|)$ when $d = 1$ and $\Lambda = [-N, N]$ for large enough N , and for some constants c and $c(\beta)$ which do not depend on Λ .

$$\mathbf{P}(\sigma) \propto \exp\left(\beta \sum_k \sigma_k \sigma_{k+1}\right) \quad \text{by symmetry on the line}$$

$$\mathbf{P}(\sigma_i | \{\sigma_j\}_{j \neq i}) \propto \exp(\beta \sigma_i (\sigma_{i-1} + \sigma_{i+1}))$$

$$\begin{aligned} \mathbf{E}_{\mu_\Lambda}(\sigma_i | \{\sigma_j\}_{j \neq i}) &= \tanh(\beta(\sigma_{i-1} + \sigma_{i+1})) \\ &= \tanh(2\beta) \frac{\sigma_{i-1} + \sigma_{i+1}}{2} \\ &= \delta \frac{\sigma_{i-1} + \sigma_{i+1}}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{E}_{\mu_\Lambda}(\sigma_i \sigma_j) &= \mathbf{E}_{\mu_\Lambda}\left(\mathbf{E}_{\mu_\Lambda}(\sigma_i | \{\sigma_j\}_{j \neq i}) \sigma_j\right) \quad \text{by the Tower property} \\ &= \frac{\delta}{2} \left(\mathbf{E}_{\mu_\Lambda}(\sigma_{i-1} \sigma_j) + \mathbf{E}_{\mu_\Lambda}(\sigma_{i+1} \sigma_j)\right) \end{aligned}$$

Iterating on the last expression to eliminate the expectation yields the exponentially decaying result.

Peierls (1936) proved that the Ising Model has a phase transition in $d \geq 2$. That is to say there exists

some β below which there is never any magnetization, and (perhaps another) β above which there is magnetization. Formally, Peierls demonstrated that $\exists \beta_0$ such that, if $\beta \geq \beta_0$ then $|\mathbf{E}_{\mu_\Lambda}(\sigma_i \sigma_j)| \geq c(\beta) > 0 \quad \forall i, j$ in any infinite volume limit (free boundary), but $\exists \beta_1 > 0$ such that, if $\beta < \beta_1$ then $|\mathbf{E}_{\mu_\Lambda}(\sigma_i \sigma_j)| \leq c \exp(-c(\beta) |i - j|) \quad \forall i, j$ and where $|i - j|$ is the Euclidean distance between i and j . Prior to illuminating Peierls' proof, we need some definitions and lemmas.

Infinite Volume Limit

Consider the Ising Model on $\Lambda \subseteq \mathbb{Z}^d$ and extend it to a probability measure on $\{-1, 1\}^{\mathbb{Z}^d}$ by defining arbitrarily outside Λ . Call this measure μ_Λ and note that $\{-1, 1\}^{\mathbb{Z}^d}$ is a compact metric space in the product topology. Taking $\Lambda_1, \Lambda_2, \dots$ as subsets which increase to \mathbb{Z}^d , $\{\mu_{\Lambda_n}\}_{n=1}^\infty$ will have a subsequential limit. Any such subsequential limit is called an infinite volume Ising model with free boundary conditions.

If f is a function of finitely many spins, then $\int f d\mu_{\Lambda_n} \rightarrow \int f d\mu_\Lambda$ as $n \uparrow \infty$ where μ_Λ is our subsequence limit.

Lemma 1

Choose any two edge sets $\eta_\Lambda^{(1)} \subseteq \eta_\Lambda^{(2)}$. Define $\eta_\Lambda^{(*)} \equiv \eta_\Lambda^{(2)} \setminus \eta_\Lambda^{(1)}$. Then define the expectation under the i.i.d. measure $\mathcal{Z}_\eta = \mathbf{E}_{\text{i.i.d.}}(\exp(\beta \sum_{i,j \in \eta} \sigma_i \sigma_j))$. We claim that

$$2^{-|\eta_\Lambda^{(1)}|} \leq \frac{\mathcal{Z}_{\eta_\Lambda^{(*)}}}{\mathcal{Z}_{\eta_\Lambda^{(2)}}} \leq 2^{|\eta_\Lambda^{(1)}|}.$$

This is proved by induction on lemma 2.

Lemma 2

For any edge set η of Λ and any edge $q = (i, j) \in \eta$ and small enough β , we have that

$$\frac{1}{2} \leq \frac{\mathcal{Z}_{\eta \setminus \{q\}}}{\mathcal{Z}_\eta} \leq 2$$

Proof:

$$\begin{aligned} \mathcal{Z}_\eta &= \mathbf{E}_{\text{i.i.d.}}\left(\exp\left(\beta \sum_{i,j \in \eta} \sigma_i \sigma_j\right)\right) \\ &= \mathbf{E}_{\text{i.i.d.}}\left(\prod_{i,j \in \eta} \exp(\beta \sigma_i \sigma_j)\right) \\ &= \mathbf{E}_{\text{i.i.d.}}\left(\prod_{e \in \eta} (1 + \rho_e)\right) \\ &= \mathbf{E}_{\text{i.i.d.}}\left((1 + \rho_q) \prod_{e \in \eta \setminus \{q\}} (1 + \rho_e)\right) \end{aligned} \quad \rho_q \equiv \exp(\beta \sigma_i \sigma_j) - 1$$

$$\begin{aligned} \mathcal{Z}_\eta - \mathcal{Z}_{\eta \setminus \{q\}} &= \mathbf{E}_{\text{i.i.d.}}\left(\left((1 + \rho_q) \prod_{e \in \eta \setminus \{q\}} (1 + \rho_e) - \prod_{e \in \eta \setminus \{q\}} (1 + \rho_e)\right)\right) \\ &= \mathbf{E}_{\text{i.i.d.}}\left(\rho_q \prod_{e \in \eta \setminus \{q\}} (1 + \rho_e)\right) \end{aligned}$$

Say that two edges are connected if they share a vertex. For $\nu \subseteq \eta$ with $q = (i, j) \in \nu$, decompose into connected components $\nu \equiv \nu_1 \cup \nu_2$ with $q \in \nu_1$ and with $\nu_2 \equiv \nu \setminus \nu_1$. Observe that $\nu \mapsto (\nu_1, \nu_2)$ is a bijection. Therefore

$$\begin{aligned}
 \mathcal{Z}_\eta - \mathcal{Z}_{\eta \setminus \{q\}} &= \sum_{(\nu_1, \nu_2)} \mathbf{E}_{\text{i.i.d.}} \left(\prod_{e \in \nu} \rho_e \right) \\
 &= \sum_{(\nu_1, \nu_2)} \mathbf{E}_{\text{i.i.d.}} \left(\prod_{e \in \nu_1} \rho_e \right) \mathbf{E}_{\text{i.i.d.}} \left(\prod_{e \in \nu_2} \rho_e \right) \\
 &= \sum_{\nu_1} \mathbf{E}_{\text{i.i.d.}} \left(\prod_{e \in \nu_1} \rho_e \right) \sum_{\nu_2 \subseteq \eta \setminus \overline{\nu_1}} \mathbf{E}_{\text{i.i.d.}} \left(\prod_{e \in \nu_2} \rho_e \right) \\
 &= \sum_{\nu_1} \mathbf{E}_{\text{i.i.d.}} \left(\prod_{e \in \nu_1} \rho_e \right) \mathcal{Z}_{\eta \setminus \overline{\nu_1}}
 \end{aligned}$$

where we defined $\overline{\nu_1} \equiv \nu_1 \cup \{\text{all edges which share a vertex with some member of } \nu_1\}$. (To be continued.)

Lecture 2: April 5

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Phase Transitions of the Ising Model

(Continued from previous lecture.) Recall that we showed

$$Z_\eta - Z_{\eta \setminus \{q\}} = \sum_{\nu_1} \mathbf{E}_{\text{i.i.d.}} \left(\prod_{e \in \nu_1} \rho_e \right) Z_{\eta \setminus \bar{\nu}_1}$$

where we defined $\bar{\nu}_1 \equiv \nu_1 \cup \{\text{all edges in } \eta \text{ which are adjacent to some edge in } \nu_1\}$. Since $|\rho_e| \leq c(\beta)$ where $c(\beta)$ is a constant that depends only on β , we get

$$\left| \frac{Z_\eta}{Z_{\eta \setminus \{q\}}} - 1 \right| \leq \sum_{\nu_1} (c(\beta))^{|\nu_1|} \frac{Z_{\eta \setminus \bar{\nu}_1}}{Z_{\eta \setminus \{q\}}}$$

Using this inequality, it is now easy to prove by induction on the size of η that if β is sufficiently small, then

$$\frac{1}{2} \leq \frac{Z_{\eta \setminus \{q\}}}{Z_\eta} \leq 2, \quad \text{as was to be shown.}$$

This is because $c(\beta) \rightarrow 0$ as $\beta \rightarrow 0$, and the number of connected sets of edges ν_1 of size k that contains q is at most C^k , where C depends only on the dimension.

Now we may prove that for sufficiently small β the expectation of any two spins being aligned is exponentially decaying in distance. Recall that if f is a function of finitely many spins, then $\int f d\mu_{\Lambda_n} \rightarrow \int f d\mu_\Lambda$ as $n \uparrow \infty$ where μ_Λ is the (free boundary) infinite volume subsequence limit of our probability density. Taking σ as some arbitrary configuration of spins for an Ising Model $\Lambda_n \subseteq \mathbb{Z}^d$, and by definition of expectation, we have that

$$\begin{aligned} \mathbf{E}_{\mu_{\Lambda_n}}(f(\sigma)) &= \frac{\mathbf{E}_{\text{i.i.d.}}(f(\sigma) \exp(\beta H(\sigma)))}{\mathbf{E}_{\text{i.i.d.}}(\exp(\beta H(\sigma)))} \\ &= \frac{1}{Z_\eta} \mathbf{E}_{\text{i.i.d.}} \left(f(\sigma) \prod_{i,j \in \eta} \exp(\beta \sigma_i \sigma_j) \right) \\ &= \frac{1}{Z_\eta} \mathbf{E}_{\text{i.i.d.}} \left(f(\sigma) \prod_{e \in \eta} (1 + \rho_e) \right) \\ &= \frac{1}{Z_\eta} \sum_{\nu \subseteq \eta} \mathbf{E}_{\text{i.i.d.}} \left(f(\sigma) \prod_{e \in \nu} \rho_e \right) \end{aligned}$$

Let \mathcal{F} be the set of vertices which determines f . Let ψ be the set of edges incident to some vertex in \mathcal{F} . Let ν_1 be the union of all *connected components* of ν that contain some element of ψ . Defining $\nu_2 \equiv \nu \setminus \nu_1$ yields

the bijection $\nu \mapsto (\nu_1, \nu_2)$. This enables the expansion of the expectation as was done in the previous lecture

$$\begin{aligned} \mathbf{E}_{\mu_{\Lambda_n}}(f(\boldsymbol{\sigma})) &= \frac{1}{Z_\eta} \sum_{(\nu_1, \nu_2)} \mathbf{E}_{\text{i.i.d.}} \left(f(\boldsymbol{\sigma}) \left(\prod_{e \in \nu_1} \rho_e \right) \left(\prod_{e \in \nu_2} \rho_e \right) \right) \\ &= \frac{1}{Z_\eta} \sum_{\nu_1} \mathbf{E}_{\text{i.i.d.}} \left(f(\boldsymbol{\sigma}) \prod_{e \in \nu_1} \rho_e \right) \sum_{\nu_2 \subseteq \eta \setminus \overline{\nu_1}} \mathbf{E}_{\text{i.i.d.}} \left(\prod_{e \in \nu_2} \rho_e \right) \\ &= \frac{1}{Z_\eta} \sum_{\nu_1} \mathbf{E}_{\text{i.i.d.}} \left(f(\boldsymbol{\sigma}) \prod_{e \in \nu_1} \rho_e \right) Z_{\eta \setminus \overline{\nu_1}} \\ &= \sum_{k=0}^{\infty} \sum_{\nu_1 : |\nu_1|=k} \mathbf{E}_{\text{i.i.d.}} \left(f(\boldsymbol{\sigma}) \prod_{e \in \nu_1} \rho_e \right) \frac{Z_{\eta \setminus \overline{\nu_1}}}{Z_\eta} \end{aligned}$$

where the number of ν_1 such that $|\nu_1| = k$ is bounded by C^k . Suppose $f(\boldsymbol{\sigma}) = \sigma_i \sigma_j$ and suppose $k < |i - j|$ then no connected component of ν_1 of size k can contain edges incident to both i and j . Such ν_1 decomposes as $\nu_1 = \nu'_1 \cup \nu''_1$ where ν'_1 is the union of all connected components of edges incident on i and ν''_1 is the union of all connected components of edges incident on j . Therefore $\nu'_1 \cap \nu''_1 = \emptyset$. By independence,

$$\mathbf{E}_{\text{i.i.d.}} \left(f(\boldsymbol{\sigma}) \prod_{e \in \nu_1} \rho_e \right) = \mathbf{E}_{\text{i.i.d.}} \left(f(\boldsymbol{\sigma}) \prod_{e \in \nu'_1} \rho_e \right) \mathbf{E}_{\text{i.i.d.}} \left(f(\boldsymbol{\sigma}) \prod_{e \in \nu''_1} \rho_e \right) = 0.$$

Therefore,

$$\begin{aligned} \mathbf{E}_{\mu_{\Lambda_n}}(f(\boldsymbol{\sigma})) &\leq \sum_{k \geq |i-j|} c(\beta)^k C^k \\ &\leq c_1 \exp(-c_2 |i-j|) \quad \text{as was to be shown.} \end{aligned}$$

Next we prove that for sufficiently large β , Ising model is magnetized in $d \geq 2$. For any $\mathcal{A} \subseteq \Lambda \subseteq \mathbb{Z}^d$ let $\mathcal{D}_{\mathcal{A}}$ be all edges with one vertex in \mathcal{A} and one vertex in $\Lambda \setminus \mathcal{A}$. Any set of edges φ will be called a *contour* if $\varphi = \mathcal{D}_{\mathcal{A}}$ for some \mathcal{A} such that \mathcal{A} and $\Lambda \setminus \mathcal{A}$ are both connected. Given some arbitrary configuration of spins $\boldsymbol{\sigma} \in \{-1, 1\}^\Lambda$, a contour φ will be called an *interface* if $\sigma_u \neq \sigma_v \quad \forall u, v \in \varphi$.

Unlikely Interfaces (Lemma 3)

Given a contour φ , the probability that φ is an interface is upper bounded by $\exp(-2\beta|\varphi|)$

Proof: Let \mathcal{X}_φ be the set of all $\boldsymbol{\sigma}$ such that φ is an interface with respect to $\boldsymbol{\sigma}$. Then

$$\mathbf{P}(\varphi \text{ is an interface}) = \frac{\sum_{\boldsymbol{\sigma} \in \mathcal{X}_\varphi} \exp(\beta H(\boldsymbol{\sigma}))}{\sum_{\boldsymbol{\sigma}} \exp(\beta H(\boldsymbol{\sigma}))}$$

Consider φ fixed and given any $\boldsymbol{\sigma} \in \mathcal{X}_\varphi$ let $T(\boldsymbol{\sigma})$ be the spin configuration obtained by flipping all the spins on one side of the contour φ , where the choice of the side is predetermined. Finally letting $\mathcal{Y}_\varphi = \text{range}(T)$ yields

$$\mathbf{P}(\varphi \text{ is an interface}) = \frac{\sum_{\boldsymbol{\sigma} \in \mathcal{X}_\varphi} \exp(\beta H(\boldsymbol{\sigma}))}{\sum_{\boldsymbol{\sigma} \in \mathcal{Y}_\varphi} \exp(\beta H(\boldsymbol{\sigma}))}$$

(To be continued.)

Lecture 3: April 10

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3.1 Nondecay of Correlations at High β for $d \geq 2$

Theorem 1 Let $d \geq 2$. Take any infinite volume limit of the Ising model on \mathbb{Z}^d at inverse temperature β . If β is sufficiently large, then there exists $C(\beta) > 0$ such that for all i, j , we have

$$\langle \sigma_i \sigma_j \rangle \geq C(\beta).$$

To prove the theorem, it suffices to consider the model in a finite connected set Λ which will eventually grow to \mathbb{Z}^d . Recall the following definitions:

Definition 2 For any $A \subset \Lambda$ let ∂A be the set of all edges with one endpoint in A and one endpoint in $\Lambda \setminus A$.

Definition 3 A set of edges γ is called a contour if $\gamma = \partial A$ for some A such that A and $\Lambda \setminus A$ are each connected.

Definition 4 Given a configuration σ , a contour γ is called an interface if $\sigma_u \neq \sigma_v$ for all $\langle uv \rangle \in \gamma$.

From the previous lecture, we have the following lemma:

Lemma 5 Given a contour γ ,

$$P(\gamma \text{ is an interface}) \leq e^{-2\beta|\gamma|}.$$

We will also need the following two facts:

Fact 6 A contour is a connected set under an appropriate graph structure on the edges of Λ .

A reference for this is Lemma 2 from Ádám Timár, "Boundary-connectivity via Graph Theory" – Proc. AMS 2013. Here's the proof from the paper:

Proof: Let \mathcal{C} be the set of 4-cycles surrounding some 2-face in a unit cube in \mathbb{Z}^d . In what follows we perform addition modulo 2 (so technically we're in the space $(\mathbb{F}_2)^{E(\Lambda)}$). This allows us to sum elements of \mathcal{C} . We remark that \mathcal{C} generates all finite cycles in \mathbb{Z}^d .

Consider the graph structure G' on the edges $E(\Lambda)$ defined as follows: It has vertex set $V(G') = E(\Lambda)$, and two edges $e = (u_1, u_2), e' = (v_1, v_2) \in E(\Lambda)$ are adjacent in G' if $d(u_i, v_j) \leq 1$ for some i, j . We will show that if $\gamma = \partial A$ with $A, \Lambda \setminus A$ each connected, then γ is a connected set in G' .

Consider any partition $\gamma = S_1 \cup S_2$ with $S_1, S_2 \neq \emptyset$ and $S_1 \cap S_2 = \emptyset$; it suffices to show that in G' we have $d(S_1, S_2) = 1$.

Arbitrarily pick $x \in A$ and $y \in \Lambda \setminus A$. Pick paths P_1, P_2 from x to y such that $P_1 \cap S_2 = \emptyset$ and $P_2 \cap S_1 = \emptyset$. Since $P_1 + P_2$ is a cycle, we may represent it as a sum

$$P_1 + P_2 = \sum_{C \in A} C \quad \text{for some finite subset } A \subset \mathcal{C}.$$

Let $A_1 \subset A$ be the elements intersecting S_1 , and $A_2 = A \setminus A_1$, so

$$P_1 + \sum_{C \in A_1} C = P_2 + \sum_{C \in A_2} C.$$

Observe that P_2 contains an odd number of terms in S_2 (since it enters and leaves A an odd number of times, and is disjoint from S_1), and each $C \in \mathcal{C}$ contains an even number of terms in S_2 . Hence RHS contains an odd number of terms in S_2 , and so LHS contains some element of S_2 ; since $P_1 \cap S_2 = \emptyset$, there exists some $O \in A_1$ which intersects S_2 . By definition this O also intersects S_1 . Since O is a 4-cycle we conclude that some edge of S_1 is within distance 1 of some edge of S_2 , and so $d(S_1, S_2) = 1$, as needed. ■

Fact 7 *If G is a graph with maximal degree Δ , then the number of connected sets of vertices of size k containing a given vertex is at most $e^{C(\Delta)k}$.*

Proof: It suffices to prove that when G is a Δ -ary tree, the number of connected k -subsets of G containing the root r is at most $e^{C(\Delta)k}$. Consider the following procedure for choosing such a subset S . Start with $S = \emptyset$ and a queue initially containing only r . At each step, we dequeue the front element v from the queue. If we choose to include v in S we enqueue its Δ children in some fixed order, and otherwise we do nothing. We stop when $|S| = k$. Clearly this procedure can give us any k -subset containing r , and moreover, any run of this procedure takes at most $k\Delta$ steps. Each step is either "keep" or "reject", and since we wish to keep precisely k vertices, there is an injection from $\{S \subset G \mid |S| = k \text{ and } r \in S\}$ into the set of all binary strings of length $k\Delta$ which contain exactly k 1's. Thus using a Stirling approximation,

$$\#\{k\text{-subsets of } G \text{ containing } r\} \leq \binom{k\Delta}{k} \leq \frac{(k\Delta)^k}{k!} \leq \frac{k^k \Delta^k}{\sqrt{2\pi k} (k/e)^k} \leq e^{C(\Delta)k}.$$

Combining these facts, we conclude that for any given edge e ,

$$\#\{\text{Contours of size } k \text{ containing } e\} \leq e^{C(d)k}.$$

The last ingredient we need is the following lemma:

Lemma 8 *If $\sigma_i \neq \sigma_j$, then there must be an interface separating i and j .*

Proof: Consider the set of all sites having the spin σ_i , and let B be the connected component of this set containing i . Let A be the connected component of $\Lambda \setminus B$ that contains j . It's easy to check that both A and $\Lambda \setminus A$ are connected. Then $\gamma = \partial A$ is the desired interface separating i and j . ■

Now we are ready to prove the theorem.

Proof: Consider $i \neq j$. We first bound the number of length- k contours that separate i and j . Observe that any such contour must pass within distance k from either i or j . There are at most Ck^d edges within distance k from either i or j , and each such edge has at most $e^{C(d)k}$ contours passing through it. Thus, defining

$$S_k = \{\text{length-}k \text{ contours separating } i \text{ and } j\},$$

we have

$$\#S_k \leq Ck^d e^{C(d)k}.$$

Now we bound the probability that $\sigma_i \neq \sigma_j$. We have

$$\begin{aligned} P(\sigma_i \neq \sigma_j) &\leq P(\text{there exists an interface separating } i, j) \\ &\leq \sum_{k=1}^{\infty} P(\text{there exists a length-}k \text{ interface separating } i, j) \\ &\leq \sum_{k=1}^{\infty} \sum_{\gamma \in S_k} P(\gamma \text{ is an interface}) \\ &\leq \sum_{k=1}^{\infty} Ck^d e^{C(d)k} e^{-2\beta k} \\ &< \frac{1}{2} \end{aligned}$$

if β is sufficiently large (size depending only on d , not on Λ or i, j).

Finally, we observe that given the above bound $P(\sigma_i \neq \sigma_j) < \frac{1}{2}$, we have that $\langle \sigma_i \sigma_j \rangle = 1 - 2P(\sigma_i \neq \sigma_j)$ is uniformly bounded away from zero. Thus correlations do not decay in the $d \geq 2$, β large regime. ■

We remark that this proof further shows that as $\beta \rightarrow \infty$, the correlations between spins goes to 1.

This theorem was important because it revealed to physicists that phase transitions can arise without large changes in dynamics.

3.2 Summary

For the Ising model, we can have shown that:

- **d = 1:** Exponential decay of correlation at all β .
- **d ≥ 2:** Exponential decay of correlation at small β , no decay at large β .

A generalization of the Ising model is the $O(n)$ model. The sites are still vertices of \mathbb{Z}^d , but instead of having spins in $\{-1, 1\}$, we have spins in S^{n-1} . The probability measure on a configuration σ is then given by the density

$$\frac{1}{Z} \exp \left(\sum_{\langle i, j \rangle} \sigma_i \cdot \sigma_j \right)$$

where $\sigma_i \cdot \sigma_j$ is the usual inner product on S^{n-1} .

- $O(1)$ is the Ising model,
- $O(2)$ is called the XY model or rotator model,
- $O(3)$ is called the XYZ model or Heisenberg model.

What is known for $O(n)$?

- Exponential decay of correlation at small β for all n, d .
- The $O(1)$ model has no decay at large β if $d \geq 2$, and has exponential decay at all β if $d = 1$.
- $O(n)$ models have no decay if $d \geq 3$ and β large. See Fröhlich, Spencer, etc's work using the method of infrared bounds.
- For $d = 2$, $O(2)$ model has decay in correlation for all β . Remarkably we have exponential decay at small β , but polynomial decay at large β . This is called the Kosterlitz-Thouless transition.
- For $d = 2$ and $n \geq 3$, it is known that polynomial decay occurs at all β . However, the Polyakov Conjecture states that exponential decay should happen at all β ! This is hard to prove because we currently have no good techniques to prove exponential decay for all β .

3.3 Exponential Decay of Correlation at Low β by Coupling

Although we've already seen a proof of the exponential decay of correlation for low β , coupling is an important technique that frequently works in proving such bounds for Markov random fields.

The Ising model is an example of a Markov Random Field. That is, the conditional distribution of $(\sigma_i)_{i \in A}$ given $(\sigma_j)_{j \in \Lambda \setminus A}$ depends only on those $j \in \Lambda \setminus A$ which are neighbors of some $i \in A$.

Theorem 9 Consider the Ising model for any d , in a box with center i and sidelength k . Fix arbitrary boundary conditions. If $\beta \leq \beta_0(d)$ then $|\langle \sigma_i \rangle| \leq C_1 e^{-C_2 k}$.

Note that this statement is much stronger than the exponential decay of correlation, and implies it as follows: Writing τ to be the conditional expectation of σ_i given all spins outside a box with center i and side length $k = |i - j|$ (so j lies outside the box), we have

$$\langle \sigma_i \sigma_j \rangle = \langle \sigma_j \tau \rangle,$$

but the theorem together with the Markov property shows that $|\tau| \leq C_1 e^{-C_2 k}$, and hence $\langle \sigma_i \sigma_j \rangle$ decays exponentially in $k = |i - j|$.

Now we prove the theorem.

Proof: Consider two different boundary conditions on the box. Let σ, η be configurations from the two Ising models with different boundary conditions. We will show that

$$|\langle \sigma_i \rangle - \langle \eta_i \rangle| \leq C_1 e^{-C_2 k} \quad \text{if } \beta \text{ is small.} \quad (3.1)$$

This proves the original claim, since we can average this relation over all boundary conditions for η , and by symmetry the average of $\langle \eta_i \rangle$ over all boundary conditions is zero.

In order to prove 3.1, we use the technique known as coupling. Coupling originated in the mid 1990s; in physics by Salas-Sokal and in concentration inequalities by Martin.

We start with independent samples $\sigma^{(0)}, \eta^{(0)}$ from the two boundary conditions. We keep updating them to get $(\sigma^{(1)}, \eta^{(1)}), (\sigma^{(2)}, \eta^{(2)}), \dots$ such that for all n ,

$$\begin{aligned} \sigma^{(n)} &\stackrel{d}{=} \sigma^{(0)}, \\ \eta^{(n)} &\stackrel{d}{=} \eta^{(0)}, \end{aligned}$$

where $X \stackrel{d}{=} Y$ means that X, Y have the same distribution. We will prove that

$$\limsup_{n \rightarrow \infty} P(\sigma_i^{(n)} \neq \eta_i^{(n)}) \leq C_1 e^{-C_2 k}, \quad (3.2)$$

which would imply 3.1 if we take the lim sup of

$$|\langle \sigma_i \rangle - \langle \eta_i \rangle| = |\langle \sigma_i^{(n)} \rangle - \langle \eta_i^{(n)} \rangle| \leq 2P(\sigma_i^{(n)} \neq \eta_i^{(n)}).$$

Updating $(\sigma^{(n)}, \eta^{(n)})$:

Choose a point $j \in \Lambda$ uniformly at random. Define

$$\mu = \text{law of } \sigma_j^{(n)} \text{ given } (\sigma_k^{(n)})_{k \neq j},$$

$$\nu = \text{law of } \eta_j^{(n)} \text{ given } (\eta_k^{(n)})_{k \neq j}.$$

Generate $(\sigma^{(n+1)}, \eta^{(n+1)})$ such that $\sigma_j^{(n+1)} \sim \mu$, $\eta_j^{(n+1)} \sim \nu$, and $P(\sigma_j^{(n+1)} \neq \eta_j^{(n+1)}) = d_{TV}(\mu, \nu)$ (see below for definition). Also update $\sigma_k^{(n+1)} = \sigma_k^{(n)}$ and $\eta_k^{(n+1)} = \eta_k^{(n)}$ for $k \neq j$.

(Proof to be completed in next lecture) ■

Definition 10 *If μ, ν are probability measures on the same σ -algebra, then the total variation distance between μ, ν is*

$$d_{TV}(\mu, \nu) = \sup_A |\mu(A) - \nu(A)|,$$

where the supremum is taken over A in the σ -algebra.

The alternative (equivalent) version of this is the following: Consider any pair of random variables (X, Y) with $X \sim \mu$ and $Y \sim \nu$. Then the minimum possible value of $P(X \neq Y)$ is $d_{TV}(\mu, \nu)$.

Lecture 4: April 12

Lecturer: Spring 2017

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4.0 Recap

Recall that we are considering the Ising model on $\Lambda \subset \mathbb{Z}^d$, which for simplicity we assume is a box with center $i \in \Lambda$ and side length k . Our goal was to prove exponential decay of spin-spin correlations at high temperatures (small β) via a coupling argument, a technique that arises frequently in probabilistic analysis. In particular, we seek to show that

$$|\langle \sigma_i \rangle| \leq C_1 e^{-C_2 k}, \quad (4.3)$$

where $\sigma \in \{\pm 1\}^\Lambda$ is a random spin configuration chosen according to an arbitrary boundary condition, and the constants C_1 and C_2 are positive and depend only on β and d . We discussed in the previous lecture that (4.3) implies exponential decay of correlations. Moreover, by an averaging principle we reasoned that to show (4.3), it suffices to prove

$$|\langle \sigma_i \rangle - \langle \eta_i \rangle| \leq C_1 e^{-C_2 k}, \quad (4.4)$$

where σ and η are random spin configurations associated to two arbitrary boundary conditions.

4.1 Proof of (4.4) at high temperatures

Let μ and ν be the probability measures on $\{\pm 1\}^\Lambda$ associated to two arbitrary (but fixed) boundary conditions. We will define a sequence of couplings of these two laws such that the marginals of the spin at site i become exponentially close in k . Here “closeness” is measured by the total variation distance between two probability measures:

$$d_{\text{TV}}(P_1, P_2) = \inf_{(X, Y) \sim (P_1, P_2)} P(X \neq Y), \quad (4.5)$$

where the infimum is over random variables X (distributed as P_1) and Y (distributed as P_2) defined on the same probability space (Ω, P) .

The coupling procedure is prescribed inductively as follows. First generate spin configurations $\sigma^{(0)} \sim \mu$ and $\eta^{(0)} \sim \nu$ independently at random. Then given $\sigma^{(n)}$ and $\eta^{(n)}$, pick a coordinate $j \in \Lambda$ uniformly at random, and let $\mu_j^{(n)}$ be the conditional distribution of $\sigma_j^{(n)}$ given $(\sigma_k^{(n)})_{k \neq j}$. Similarly, let $\nu_j^{(n)}$ denote the conditional distribution of $\eta_j^{(n)}$ given $(\eta_k^{(n)})_{k \neq j}$. We will update the spins at site j to $\sigma_j^{(n+1)}$ and $\eta_j^{(n+1)}$ by sampling from $\mu_j^{(n)}$ and $\nu_j^{(n)}$, respectively, but *not independently*. Spins at all other sites $k \neq j$ are unchanged. Having described the update procedure, let us pause to make some observations:

Observation 11 *The conditional distributions under consideration are probabilities on $\{\pm 1\}$, so there surely exists a coupling of $\mu_j^{(n)}$ and $\nu_j^{(n)}$ on some (Ω, P) such that $P(\sigma_j^{(n+1)} \neq \eta_j^{(n+1)})$ achieves the infimum in the*

definition (4.5). That is, we may generate the pair $(\sigma_j^{(n+1)}, \eta_j^{(n+1)})$ such that the new spins are equal with the highest possible probability:

$$P(\sigma_j^{(n+1)} \neq \eta_j^{(n+1)}) = d_{\text{TV}}(\mu_j^{(n)}, \nu_j^{(n)}). \quad (4.6)$$

For example, if the marginals must satisfy $P(\sigma_j^{(n+1)} = +1) = 0.3$ and $P(\eta_j^{(n+1)} = +1) = 0.5$, then we simply choose a coupling so that

$$P(\sigma_j^{(n+1)} = a, \eta_j^{(n+1)} = b) = \begin{cases} 0.3 & (a, b) = (+1, +1) \\ 0.5 & (a, b) = (-1, -1) \\ 0.2 & (a, b) = (-1, +1) \\ 0 & (a, b) = (+1, -1). \end{cases}$$

Observation 12 The conditional distributions actually depend only on the spins at neighbors of j , a set we will write as $\mathcal{N}(j) \subset \Lambda$. In particular, if $\sigma_k^{(n)} = \eta_k^{(n)}$ for all $k \in \mathcal{N}(j)$, then

$$d_{\text{TV}}(\mu_j^{(n)}, \nu_j^{(n)}) = 0. \quad (4.7)$$

Observation 13 When $\beta = 0$, regardless of neighboring spins, the conditional distribution of the spin at site j is uniform over $\{\pm 1\}$, provided $j \notin \partial\Lambda$. So (4.7) always holds in this case. For $\beta > 0$, there is still an upper bound C depending only on β and d :

$$d_{\text{TV}}(\mu_j^{(n)}, \nu_j^{(n)}) \leq C, \quad (4.8)$$

again provided that $j \notin \partial\Lambda$. Moreover, $C \rightarrow 0$ as $\beta \rightarrow 0$. (Exercise: Check that C can be taken equal to $\tanh(2d\beta)$.)

Observation 14 Since $\sigma_j^{(n+1)}$ has the same distribution as $\sigma_j^{(n)}$, and all other spins are unchanged, the law of $\sigma^{(n+1)}$ is the same as that of $\sigma^{(n)}$. Therefore, $\sigma^{(n)} \sim \mu$ for every n . By the same logic, $\eta^{(n)} \sim \nu$ for every n . Therefore,

$$\begin{aligned} |\langle \sigma_i \rangle - \langle \eta_i \rangle| &= |2\mathbf{P}(\sigma_i^{(n)} = 1, \eta_i^{(n)} = -1) - 2\mathbf{P}(\sigma_i^{(n)} = -1, \eta_i^{(n)} = 1)| \\ &\leq 2\mathbf{P}(\sigma_i^{(n)} \neq \eta_i^{(n)}) \quad \text{for all } n \\ \Rightarrow |\langle \sigma_i \rangle - \langle \eta_i \rangle| &\leq 2 \limsup_{n \rightarrow \infty} \mathbf{P}(\sigma_i^{(n)} \neq \eta_i^{(n)}). \end{aligned} \quad (4.9)$$

Now let $N = |\Lambda|$. At the n -th update, there is a $1/N$ probability that j is picked as the update site. If j is not picked, then trivially we have

$$\mathbf{P}(\sigma_j^{(n+1)} \neq \eta_j^{(n+1)} \mid \sigma^{(n)}, \eta^{(n)}, j \text{ not picked}) = \mathbf{1}_{\{\sigma_j^{(n)} \neq \eta_j^{(n)}\}}.$$

On the other hand, if j is picked and $j \notin \partial\Lambda$, then

$$\begin{aligned} \mathbf{P}(\sigma_j^{(n+1)} \neq \eta_j^{(n+1)} \mid \sigma^{(n)}, \eta^{(n)}, j \text{ picked}) &\stackrel{(4.6)}{=} d_{\text{TV}}(\mu_j^{(n)}, \nu_j^{(n)}) \\ &\stackrel{(4.7)}{=} d_{\text{TV}}(\mu_j^{(n)}, \nu_j^{(n)}) \mathbf{1}_{\{\sigma_k^{(n)} \neq \eta_k^{(n)} \text{ for some } k \in \mathcal{N}(j)\}} \\ &\stackrel{(4.8)}{\leq} C \mathbf{1}_{\{\sigma_k^{(n)} \neq \eta_k^{(n)} \text{ for some } k \in \mathcal{N}(j)\}} \\ &\leq C \sum_{k \in \mathcal{N}(j)} \mathbf{1}_{\{\sigma_k^{(n)} \neq \eta_k^{(n)}\}}. \end{aligned}$$

Finally, if j is picked but $j \in \partial\Lambda$, then no updates can be made because of the boundary conditions. Consequently, the probability of equality is 0 or 1, depending on if the two boundary conditions agree at site j :

$$\mathbf{P}(\sigma_j^{(n+1)} \neq \eta_j^{(n+1)} \mid \sigma^{(n)}, \eta^{(n)}, j \text{ picked}) = \mathbb{1}_{\{\sigma_j^{(n)} \neq \eta_j^{(n)}\}}.$$

The three cases we just considered can be expressed in a single inequality:

$$\mathbf{P}(\sigma_j^{(n+1)} \neq \eta_j^{(n+1)} \mid \sigma^{(n)}, \eta^{(n)}) \leq \left(1 - \frac{1}{N}\right) \mathbb{1}_{\{\sigma_j^{(n)} \neq \eta_j^{(n)}\}} + \frac{C}{N} \sum_{k \in \mathcal{N}(j)} \mathbb{1}_{\{\sigma_k^{(n)} \neq \eta_k^{(n)}\}} + \frac{h_j}{N},$$

where

$$h_j := \begin{cases} 1 & j \in \partial\Lambda \\ 0 & \text{else.} \end{cases}$$

Averaging over all possible realizations of $\sigma^{(n)}$ and $\eta^{(n)}$, we find

$$\mathbf{P}(\sigma_j^{(n+1)} \neq \eta_j^{(n+1)}) \leq \left(1 - \frac{1}{N}\right) \mathbf{P}(\sigma_j^{(n)} \neq \eta_j^{(n)}) + \frac{C}{N} \sum_{k \in \mathcal{N}(j)} \mathbf{P}(\sigma_k^{(n)} \neq \eta_k^{(n)}) + \frac{h_j}{N}. \quad (4.10)$$

We have derived a recursive inequality (4.10). Given the interaction between neighboring sites, it is useful to express this inequality in terms of matrix-vector products. Upon defining the vectors

$$x^{(n)} := (x_j^{(n)})_{j \in \Lambda}, \quad x_j^{(n)} := \mathbf{P}(\sigma_j^{(n)} \neq \eta_j^{(n)}), \\ h := (h_j)_{j \in \Lambda},$$

and the matrix

$$Q := (q_{jk})_{j,k \in \Lambda}, \quad q_{jk} := \begin{cases} C & k \in \mathcal{N}(j) \\ 0 & \text{else,} \end{cases}$$

we can write the coordinate-wise inequality

$$x^{(n+1)} \leq \left(1 - \frac{1}{N}\right) x^{(n)} + \frac{1}{N} Q x^{(n)} + \frac{h}{N}.$$

As the above display holds for every n , we in fact have

$$x \leq \left(1 - \frac{1}{N}\right) x + \frac{1}{N} Q x + \frac{h}{N},$$

where

$$x_j := \limsup_{n \rightarrow \infty} x_j^{(n)}, \quad j \in \Lambda.$$

Rearranging terms gives

$$(I - Q)x \leq h. \quad (4.11)$$

Recall the definition of the matrix norm

$$\|Q\|_{\infty \rightarrow \infty} := \sup\{\|Qz\|_{\infty} : \|z\|_{\infty} \leq 1\}.$$

Since each row of Q has at most $2d$ nonzero entries,

$$\|Q\|_{\infty \rightarrow \infty} \leq 2dC.$$

Here, for the first and only time, we make an assumption on β . Namely, we may assume by Observation 13 that β is sufficiently small so that

$$2dC < 1. \quad (4.12)$$

It follows that if $R := \sum_{m=0}^{\infty} Q^m$, then

$$\|R\|_{\infty \rightarrow \infty} \leq \sum_{m=0}^{\infty} \|Q^m\|_{\infty \rightarrow \infty} \leq \sum_{m=0}^{\infty} \|Q\|_{\infty \rightarrow \infty}^m < \infty.$$

Furthermore,

$$R(I - Q) = R - RQ = R - \sum_{m=0}^{\infty} Q^{m+1} = R - (R - I) = I.$$

Since all entries of R are nonnegative, (4.11) implies

$$x \leq Rh = \sum_{m=0}^{\infty} Q^m h,$$

where the j -th entry of $Q^m h$ is

$$(Q^m h)_j = \sum_{j_1, j_2, \dots, j_m} q_{jj_1} q_{j_1 j_2} \cdots q_{j_{m-1} j_m} h_{j_m}.$$

Notice that the product $q_{jj_1} q_{j_1 j_2} \cdots q_{j_{m-1} j_m} h_{j_m}$ is zero if either $j_m \in \partial\Lambda$ or (j, j_1, \dots, j_m) is not a lattice path. In particular, when $m < k/2$ and $j = i$ is at the center of Λ (which has side length k), at least one of these two statements must be true. Hence

$$|\langle \sigma_i \rangle - \langle \eta_i \rangle| \stackrel{(4.9)}{\leq} 2x_i \leq 2 \sum_{m \geq k/2} (Q^m h)_i \leq 2 \sum_{m \geq k/2} \|Q^m h\|_{\infty} \leq 2 \sum_{m \geq k/2} \|Q\|_{\infty \rightarrow \infty}^m \|h\|_{\infty} \leq 2 \frac{(2dC)^{k/2}}{1 - 2dC}.$$

Setting our constants to be

$$C_1 := \frac{2}{1 - 2dC}, \quad C_2 := -\frac{1}{2} \log(2dC),$$

we have proved (4.4) for β such that (4.12) holds.

4.2 $O(2)$ model: polynomial decay for low temperatures in $d = 2$

Recall the $O(n)$ model in which spins assume values in the sphere \mathbb{S}^{n-1} , and the Hamiltonian is the sum of inner products between neighboring spins:

$$H(\sigma) := \sum_{\langle j k \rangle} \sigma_j \cdot \sigma_k$$

In this section, we begin a proof (due to McBryan and Spencer [2]) that spin-spin correlations in two-dimensional $O(n)$ models decay faster than a polynomial for sufficiently large β . More precisely, for any $n \geq 2$, for all sufficiently large β ,

$$\langle \sigma_j \cdot \sigma_k \rangle \leq C_1 |j - k|^{-C_2},$$

where C_1 and C_2 are positive constants depending only on β and n . We will consider just the case $n = 2$, also known as the XY model.

Remark 15 *In the previous lecture we mentioned that correlations decay exponentially for small β . It is interesting, then, that the proof we present works only for large β . Although, the Kosterlitz-Thouless transition [1] says that for $n = 2$, exponential decay does not hold for large β . Meanwhile, for $n \geq 3$, Polyakov [3] conjectured that exponential decay holds for all temperatures.*

Remark 16 *The result under discussion is a special case of the Mermin-Wagner theorem in physics, which roughly says, "Systems with continuous symmetries have no ordered phases in dimension two." Here the term ordered means having long-range interactions. Indeed, since we are considering $n \geq 2$, the spins in \mathbb{S}^{n-1} have continuous symmetry.*

The setting we consider is $\Lambda \subset \mathbb{Z}^2$. A spin configuration will be thought of as an element of $[-\pi, \pi]^\Lambda$, and for simplicity we assume all boundary spins are 0. With this parameterization, the Hamiltonian is written

$$H(\theta) = \sum_{\langle j, k \rangle} \cos(\theta_j - \theta_k).$$

The correlation we wish to study is

$$\langle \cos(\theta_u - \theta_v) \rangle,$$

for $u, v \in \Lambda \setminus \partial\Lambda$.

Claim 17 *If β is large enough, then*

$$|\langle \cos(\theta_u - \theta_v) \rangle| \leq C_1 |u - v|^{-C_2(\beta)}.$$

We now begin the proof. To simplify notation, denote $\overset{\circ}{\Lambda} := \Lambda \setminus \partial\Lambda$. By symmetry, we must have $\langle \sin(\theta_u - \theta_v) \rangle = 0$, and so

$$\langle \cos(\theta_u - \theta_v) \rangle = \langle e^{i(\theta_u - \theta_v)} \rangle = \frac{1}{Z} \int_{[-\pi, \pi]^{\overset{\circ}{\Lambda}}} e^{i(\theta_u - \theta_v)} e^{\beta H(\theta)} d\theta,$$

where $d\theta$ denotes the usual Lebesgue integration, and the normalizing constant

$$Z = \int_{[-\pi, \pi]^{\overset{\circ}{\Lambda}}} e^{\beta H(\theta)} d\theta$$

makes the integral a valid expectation.

Let $a = (a_j)_{j \in \Lambda}$ be any vector of real numbers such that $a_j = 0$ for all $j \in \partial\Lambda$ (later we will choose the other values in an optimal way). We can order the elements of $\overset{\circ}{\Lambda}$ as j_1, j_2, \dots, j_N so that

$$\int_{[-\pi, \pi]^{\overset{\circ}{\Lambda}}} e^{i(\theta_u - \theta_v)} e^{\beta H(\theta)} d\theta = \int_{[-\pi, \pi]} \dots \int_{[-\pi, \pi]} e^{i(\theta_u - \theta_v)} e^{\beta H(\theta)} d\theta_1 \dots d\theta_N.$$

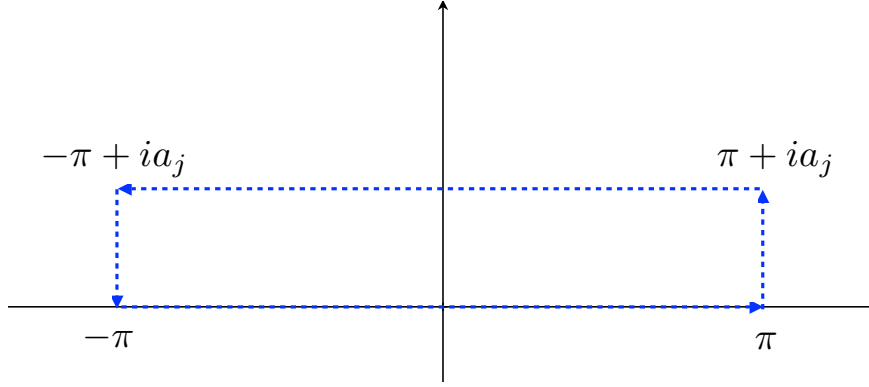


Figure 4.1: Contour of integration

For each j (starting at j_1 , then j_2 , and so on), we can rewrite the integral over $[-\pi, \pi)$ as the same integral over the line segment $[-\pi + ia_j, \pi + ia_j)$. To see this, one can consider the same integrand, but integrated over the closed contour shown in Figure 4.1. Notice that the integrals along the vertical segments will cancel one another, since

$$e^{i(\pi + ia_u - \theta_v)} = e^{2\pi i} e^{i(-\pi + ia_u - \theta_v)} = e^{i(-\pi + ia_u - \theta_v)},$$

and similarly

$$\cos(\pi + ia_j - \theta_k) = \cos(-\pi + ia_j - \theta_k),$$

for any θ_v, θ_k (real or complex). Therefore, by Cauchy's integral formula, the integral over $[-\pi, \pi)$ is equal to the integral over the line segment $[-\pi + ia_j, \pi + ia_j)$. As claimed, then, we can write

$$\langle \cos(\theta_u - \theta_v) \rangle = \frac{1}{Z} \int_{[-\pi, \pi)^{\wedge}} e^{i(\theta_u + ia_u - \theta_v - ia_v)} e^{\beta H(\theta + ia)} d\theta.$$

Upon recalling the identity

$$\begin{aligned} \cos(x + iy) &= \cos(x) \cos(iy) - \sin(x) \sin(iy) \\ &= \cos(x) \cosh(y) - i \sin(x) \sinh(y), \end{aligned}$$

we can expand the Hamiltonian as

$$\begin{aligned} H(\theta + ia) &= \sum_{\langle j k \rangle} \cos(\theta_j + ia_j - \theta_k - ia_k) \\ &= \sum_{\langle j k \rangle} \cos(\theta_j - \theta_k) \cosh(a_j - a_k) - i \sum_{\langle j k \rangle} \sin(\theta_j - \theta_k) \sinh(a_j - a_k). \end{aligned}$$

Hence

$$\begin{aligned} |e^{\beta H(\theta + ia)}| &= \exp\left(\beta \sum_{\langle j k \rangle} \cos(\theta_j - \theta_k) \cosh(a_j - a_k)\right) \\ &= \exp\left(\beta \sum_{\langle j k \rangle} \cos(\theta_j - \theta_k) (\cosh(a_j - a_k) - 1)\right) \exp\left(\beta \sum_{\langle j k \rangle} \cos(\theta_j - \theta_k)\right) \\ &\leq \exp\left(\beta \sum_{\langle j k \rangle} (\cosh(a_j - a_k) - 1)\right) e^{\beta H(\theta)}, \end{aligned}$$

where the inequality is justified by the fact that $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$. Using these observations, we find

$$\begin{aligned} |\langle \cos(\theta_u - \theta_v) \rangle| &\leq \frac{1}{Z} \int_{[-\pi, \pi]^{\hat{\Lambda}}} |e^{i(\theta_u + ia_u - \theta_v - ia_v)} e^{\beta H(\theta + ia)}| \, d\theta \\ &\leq \frac{1}{Z} \exp\left(- (a_u - a_v) + \beta \sum_{\langle j k \rangle} (\cosh(a_j - a_k) - 1)\right) \int_{[-\pi, \pi]^{\hat{\Lambda}}} e^{\beta H(\theta)} \, d\theta \\ &= \exp\left(- (a_u - a_v) + \beta \sum_{\langle j k \rangle} (\cosh(a_j - a_k) - 1)\right). \end{aligned}$$

In the next lecture we will choose the a_j to minimize this upper bound.

References

- [1] John Michael Kosterlitz and David James Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *Journal of Physics C: Solid State Physics*, 6(7):1181, 1973.
- [2] Oliver A McBryan and Thomas Spencer. On the decay of correlations in so (n)-symmetric ferromagnets. *Communications in Mathematical Physics*, 53(3):299–302, 1977.
- [3] Alexander M Polyakov. Interaction of goldstone particles in two dimensions. applications to ferromagnets and massive yang-mills fields. *Physics Letters B*, 59(1):79–81, 1975.

Lecture 5: April 17

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Disclaimer: *These notes may contain factual and/or typographic errors.*

5.3 Continuation: Polynomial decay of correlation at low temperature in 2-dimensional $O(2)$ model

Let's recall some notation. We have the collection of sites $\Lambda \subset \mathbb{Z}^2$, with spins $\theta = (\theta_u)_{u \in \Lambda} \in [-\pi, \pi]^\Lambda$. We imposed the boundary condition $\theta_u = 0$ for $u \in \partial\Lambda$.

Recall from the last lecture that, given any $u, v \in \Lambda$ and any choice of real numbers $(a_j)_{j \in \Lambda}$ with $a_j = 0$ on $\partial\Lambda$, we have

$$\langle \cos(\theta_u - \theta_v) \rangle \leq \exp \left(-(a_u - a_v) + \frac{\beta}{2} \sum_{\langle j k \rangle} (\cosh(a_j - a_k) - 1) \right).$$

We want to choose the $(a_j)_{j \in \Lambda}$ so that the bound is strong. We reparametrize, writing $b_j = \beta a_j$. We'll choose $(b_j)_{j \in \Lambda} \in \mathbb{R}^\Lambda$ to be the solution to

$$\begin{aligned} \underset{b}{\text{minimize}} \quad & -(b_u - b_v) + \frac{1}{2} \sum_{\langle j k \rangle} (b_j - b_k)^2 \\ \text{subject to} \quad & b_j = 0, \quad j \in \partial\Lambda. \end{aligned}$$

Recall the standard definition of the discrete Laplacian operator

$$(\Delta b)_j = \sum_{k \in N(j)} (b_k - b_j), \quad \text{where } N(j) \text{ is the set of vertices adjacent to } j.$$

Taking finite differences, the minimizer of the above system satisfies

$$(\Delta b)_j = \begin{cases} 0 & \text{if } j \neq u, v \\ 1 & \text{if } j = u, \\ -1 & \text{if } j = v. \end{cases}$$

We can invert this using the discrete Green's function, to get

$$b_j = \sum_k G(j, k) l_k,$$

where

$$l_u = 1, l_v = -1, l_k = 0 \text{ for all other } k.$$

Substituting this into the given expression, we see that the minimizer of the above system is

$$b_j = G(j, u) - G(j, v).$$

We will see later in the course that

$$b_u - b_v = G(u, u) + G(v, v) - 2G(u, v) \approx C \log |u - v|.$$

Moreover, it's well known that at this optimal $(b_j)_{j \in \Lambda}$ we have

$$\sum_{\langle j k \rangle} (b_j - b_k)^2 = b_u - b_v.$$

Given a uniform bound $|b_j - b_k| \leq C$ for neighboring j, k , we get a corresponding bound $|a_j - a_k| \leq \frac{C}{\beta}$, and so a Taylor expansion of cosh yields

$$\cosh(a_j - a_k) - 1 \leq \frac{1 + \epsilon}{2} (a_j - a_k)^2,$$

where $\epsilon = \epsilon(\beta) \rightarrow 0$ as $\beta \rightarrow \infty$. Choose β sufficiently large so that $\epsilon < 1$, then

$$\begin{aligned} -(a_u - a_v) + \frac{\beta}{2} \sum_{\langle j k \rangle} (\cosh(a_j - a_k) - 1) &\leq -(a_u - a_v) + \frac{\beta}{2} \sum_{\langle j k \rangle} (a_j - a_k)^2 \\ &= -(a_u - a_v) + \frac{1}{2\beta} \sum_{\langle j k \rangle} (b_j - b_k)^2 \\ &= -(a_u - a_v) + \frac{1}{2\beta} (b_u - b_v) \\ &= -(a_u - a_v) + \frac{1}{2} (a_u - a_v) \\ &= -\frac{1}{2} (a_u - a_v) \\ &\approx -\frac{C}{2\beta} \log |u - v|. \end{aligned}$$

Returning to our original inequality, we finally get

$$\langle \cos(\theta_u - \theta_v) \rangle \leq \exp \left(-(a_u - a_v) + \frac{\beta}{2} \sum_{\langle j k \rangle} (\cosh(a_j - a_k) - 1) \right) \leq |u - v|^{-\frac{C}{2\beta}}.$$

5.4 Existence of ordered phase in $O(n)$ models in $d \geq 3$ (Fröhlich, Simon, Spencer)

For simplicity, we work in the discrete torus $\mathbb{T}_L^d = \{-L + 1, -L + 2, \dots, L - 1, L\}^d$. Our spin configuration σ lies in $(S^{n-1})^{\mathbb{T}_L^d}$, and $H(\sigma) = \sum_{\langle uv \rangle} \sigma_u \cdot \sigma_v$. For convenience, write $V = V(\mathbb{T}_L^d)$.

Theorem 18

$$\frac{1}{|V|^2} \sum_{x, y \in V} \langle \sigma_x \cdot \sigma_y \rangle \geq C_{n, d, \beta} > 0 \quad \text{if } d \geq 3 \text{ and } \beta \text{ sufficiently large.}$$

Crucially, this lower bound $C_{n, d, \beta}$ is independent of L .

This proof will depend on three important concepts: Gaussian domination, the infrared bound, and reflection positivity.

Let \mathbb{E} be the expectation with respect to the iid measure on $(S^{n-1})^\Lambda$ (that is, when we take $\beta = 0$). For any function $\tau : V \rightarrow \mathbb{R}^n$, let

$$Z(\tau) = \mathbb{E} \exp \left(-\frac{\beta}{2} \sum_{\langle u v \rangle} \|\sigma_u + \tau_u - \sigma_v - \tau_v\|^2 \right),$$

where we understand the spin $\sigma_u \in S^{n-1}$ to be a unit vector in \mathbb{R}^n .

Lemma 19 (Gaussian domination)

$$Z(\tau) \leq Z(0) \quad \text{for all } \tau : V \rightarrow \mathbb{R}^n.$$

Lemma 20 (Infrared bound)

$$\frac{1}{|V|} \sum_{u,v \in V} e^{ik \cdot (v-u)} \langle \sigma_u \cdot \sigma_v \rangle \leq \frac{n}{2\beta \sum_{j=1}^n (1 - \cos k_j)}$$

where $k = (k_j)_{j=1}^n \in V^* \setminus \{0\}$ and $V^* = \frac{\pi}{L} V$.

We note that by translation invariance of the torus, letting 0 denote the origin,

$$\frac{1}{|V|} \sum_{u,v \in V} e^{ik \cdot (v-u)} \langle \sigma_u \cdot \sigma_v \rangle = \frac{1}{|V|} \sum_{v \in V} e^{ik \cdot v} \langle \sigma_0 \cdot \sigma_v \rangle = \widehat{f}(k)$$

is a Fourier transform.

The proof of the theorem uses the infrared bound, and the proof of the infrared bound depends on Gaussian domination. Gaussian domination is proved via reflection positivity.

We can write $V = V_0 \cup V_1$, where

$$V_0 = \{(x_1, \dots, x_d) \in V : x_1 \leq 0\},$$

$$V_1 = \{(x_1, \dots, x_d) \in V : x_1 \geq 1\}.$$

Let $\theta(x_1, \dots, x_d) = (-x_1 + 1, x_2, \dots, x_d)$ be the reflection across the plane $x_1 = \frac{1}{2}$. We can write any $\tau : V \rightarrow \mathbb{R}^n$ as $\tau = (\tau_0, \tau_1)$, with $\tau_i : V_i \rightarrow \mathbb{R}^n$ for $i = 0, 1$.

For $\tau_0 : V_0 \rightarrow \mathbb{R}^n$, let $\theta(\tau_0) : V_1 \rightarrow \mathbb{R}^n$ be defined by

$$\theta(\tau_0)(x) = \tau_0(\theta(x)).$$

Likewise, for $\tau_1 : V_1 \rightarrow \mathbb{R}^n$, define $\theta(\tau_1) : V_0 \rightarrow \mathbb{R}^n$.

Lemma 21 (Reflection positivity) For all τ_0, τ_1 ,

$$Z(\tau_0, \tau_1)^2 \leq Z(\tau_0, \theta(\tau_0))Z(\theta(\tau_1), \tau_1).$$

Equipped with this, we can prove Gaussian domination:

Proof:[Gaussian domination] We first prove that there exists some τ maximizing $Z(\tau)$. Clearly Z is bounded above by 1, so $\sup_\tau Z(\tau)$ exists. Let τ^1, τ^2, \dots be a sequence that approaches $\sup_\tau Z(\tau)$. Since $Z(\tau) =$

$Z(\tau + c)$ for any constant c , we may assume that $\tau_0^n = 0$ for all n . We claim that there exists some C such that $\tau_v^n < C$ for all $n \in \mathbb{N}, v \in V$. Indeed, if not, then there exists a sequence $\tau_{v_i}^{n_i}$ which grows to infinity, then since $\tau_0^{n_i} = 0$ and $\|\sigma_{v_i}\| = \|\sigma_0\| = 1$, we have $\|\sigma_{v_i} + \tau_{v_i}^{n_i} - \sigma_0 - \tau_0^{n_i}\| \rightarrow \infty$ also. Then $Z(\tau^{n_i}) \rightarrow 0$, a contradiction. Thus $\tau_v^n < C$ for all $n \in \mathbb{N}, v \in V$, so by compactness there exists some increasing sequence n_1, n_2, \dots such that τ^{n_i} converges pointwise to some τ^∞ . Since Z is continuous in τ , we conclude that $Z(\tau^\infty) = \sup_\tau Z(\tau)$. Hence some function τ attains the maximal value of Z .

For any τ , let

$$k(\tau) = \#\{\langle u v \rangle : \tau_u \neq \tau_v\}$$

be the number of pairs of adjacent sites whose τ -values disagree. Among all τ maximizing Z , pick τ minimizing $k(\tau)$. We claim that $k(\tau) = 0$. Suppose otherwise; let $\langle u v \rangle$ be one edge for which $\tau_u \neq \tau_v$. Without loss of generality (by translation invariance), assume $u \in V_0$ and $v \in V_1$.

Let $\tau = (\tau_0, \tau_1)$, and define $\tau' = (\tau_0, \theta(\tau_0))$ and $\tau'' = (\theta(\tau_1), \tau_1)$. Observe that performing the reflection θ removes the disagreement at the edge $\langle u v \rangle$, so either $k(\tau) > k(\tau')$ or $k(\tau) > k(\tau'')$; WLOG it's the former.

By reflection positivity, we have

$$Z(\tau)^2 \leq Z(\tau')Z(\tau''),$$

and since $Z(\tau)$ maximizes Z , we conclude that $Z(\tau') = Z(\tau'') = Z(\tau)$. Recall however that $k(\tau') < k(\tau)$, contradicting how τ was chosen.

Thus, the constant function is a maximizer of Z . Since $Z(\tau) = Z(\tau + c)$ for any constant c , we conclude that $Z(\tau) \leq Z(0)$ for all τ .

■

Lecture 6: April 19

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Disclaimer: These notes may contain factual and/or typographic errors.

6.0 Recap

We are considering the $O(n)$ model in dimension $d \geq 3$, where $n \geq 1$ is arbitrary. To simplify analysis, we defined on the model on the torus with side length $2L$:

$$\mathbb{T}_L^d = \{-L + 1, \dots, L\}^d,$$

with edge set $E = E(\mathbb{T}_L^d)$ and vertex set $V = V(\mathbb{T}_L^d)$. Spin configurations $\sigma \in (\mathbb{S}^{n-1})^{\mathbb{T}_L^d}$ are subject to the Hamiltonian

$$H(\sigma) = \sum_{\langle uv \rangle \in E} \sigma_u \cdot \sigma_v.$$

In the last lecture, we stated some preliminary lemmas with the goal of proving the following result.

Theorem 22 For any $d \geq 3$, and all β and L sufficiently large (depending on d), there exists a constant $C = C(n, d, \beta) > 0$ such that

$$\frac{1}{|V|^2} \sum_{x, y \in V} \langle \sigma_x \cdot \sigma_y \rangle \geq C.$$

Let us recall some notation. We considered a partition of $V = V_0 \cup V_1$, where

$$V_0 := \{(x_1, \dots, x_d) \in V : x_1 \leq 0\}$$

$$V_1 := \{(x_1, \dots, x_d) \in V : x_1 \geq 1\},$$

along with a reflection map $\theta : V \rightarrow V$ sending vectors between the two sets:

$$\theta(x_1, x_2, \dots, x_d) = (1 - x_1, x_2, \dots, x_d).$$

Given a map $\tau : V \rightarrow \mathbb{R}^n$, we denoted its restrictions $\tau_0 = \tau|_{V_0}$ and $\tau_1 = \tau|_{V_1}$, and made the identification $\tau = (\tau_0, \tau_1)$. Moreover, given $\tau_0 : V_0 \rightarrow \mathbb{R}^n$, we defined the function $\theta(\tau_0) : V_1 \rightarrow \mathbb{R}^n$ by

$$\theta(\tau_0)(x) := \tau_0(\theta(x)), \quad x \in V_1.$$

Analogously, for $\tau_1 : V_1 \rightarrow \mathbb{R}^n$, we have

$$\theta(\tau_1)(x) := \tau_1(\theta(x)), \quad x \in V_0.$$

The key quantity we started to examine is

$$Z(\tau) := \mathbf{E} \exp \left(-\frac{\beta}{2} \sum_{\langle uv \rangle \in E} \|\sigma_u + \tau_u - \sigma_v - \tau_v\|^2 \right),$$

where $\mathbf{E}(\cdot)$ denotes expectation with respect to the i.i.d. measure on spins. We stated but did not prove the following lemma, which resembles a Cauchy-Schwarz inequality.

Lemma 23 (Reflection positivity) For any $\tau = (\tau_0, \tau_1)$,

$$Z(\tau)^2 \leq Z(\tau_0, \theta(\tau_0)) \cdot Z(\theta(\tau_1), \tau_1).$$

Assuming this result, we did prove the next lemma. (The name comes from the fact that a Gaussian system would give equality.)

Lemma 1 (Gaussian domination) For any $\tau : V \rightarrow \mathbb{R}^n$,

$$Z(\tau) \leq Z(0).$$

Finally, we stated without proof the so-called “infrared bound”, which gives a uniform bound on the nonzero Fourier coefficients of the spin correlation function $v \mapsto \langle \sigma_0 \cdot \sigma_v \rangle$.

Lemma 2 (Infrared bound) Let $V^* = (\pi/L)V$. For any $k \in V^* \setminus \{0\}$,

$$\frac{1}{|V|} \sum_{u,v \in V} e^{ik \cdot (v-u)} \langle \sigma_u \cdot \sigma_v \rangle \leq \frac{n}{2\beta \sum_{j=1}^d (1 - \cos k_j)}.$$

6.1 Proof of Lemma 1a

Fix $\tau : V \rightarrow \mathbb{R}^n$. Let $\sigma : V \rightarrow \mathbb{S}^{n-1} \subset \mathbb{R}^n$ be a spin configuration such that $(\sigma_u)_{u \in V}$ are i.i.d. uniform random variables on \mathbb{S}^{n-1} . Consider the edges bridging V_0 and V_1 :

$$E' := \{\langle uv \rangle : u \in V_0, v \in V_1\}.$$

In this notation,

$$Z(\tau) = Z(\tau_0, \tau_1) = \mathbf{E} \exp \left(f_0(\sigma_0, \tau_0) + f_1(\sigma_1, \tau_1) - \frac{\beta}{2} \sum_{\langle uv \rangle \in E'} \|\sigma_u + \tau_u - \sigma_v - \tau_v\|^2 \right),$$

where we have separated the contributions from just V_0 or just V_1 :

$$\begin{aligned} f_0(\sigma_0, \tau_0) &= -\frac{\beta}{2} \sum_{\langle uv \rangle \in E(V_0)} \|\sigma_u + \tau_u - \sigma_v - \tau_v\|^2 \\ f_1(\sigma_1, \tau_1) &= -\frac{\beta}{2} \sum_{\langle uv \rangle \in E(V_1)} \|\sigma_u + \tau_u - \sigma_v - \tau_v\|^2. \end{aligned}$$

This will be a convenient representation after we apply a trick sometimes called “Gaussian disintegration” to linearize the exponent. Recall that if $X \sim \mathcal{N}(0, \mathbf{I})$ is a standard Gaussian random vector in \mathbb{R}^n , then its Fourier transform is

$$\mathbf{E}_X(e^{i\theta \cdot X}) = e^{-\|\theta\|^2/2}, \quad \theta \in \mathbb{R}^n. \quad (6.13)$$

Hence

$$\begin{aligned}
\mathbf{E} \exp \left(-\frac{\beta}{2} \sum_{\langle u v \rangle \in E'} \|\sigma_u + \tau_u - \sigma_v - \tau_v\|^2 \right) &= \mathbf{E} \left[\prod_{\langle u v \rangle \in E'} \exp \left(-\frac{\beta}{2} \|\sigma_u + \tau_u - \sigma_v - \tau_v\|^2 \right) \right] \\
&\stackrel{(6.13)}{=} \mathbf{E} \left[\prod_{\langle u v \rangle \in E'} \mathbf{E}_{X_{uv}} \exp(i\sqrt{\beta}(\sigma_u + \tau_u - \sigma_v - \tau_v) \cdot X_{uv}) \right] \\
&= \mathbf{E} \left[\mathbf{E}_X \left(\prod_{\langle u v \rangle \in E'} \exp(i\sqrt{\beta}(\sigma_u + \tau_u - \sigma_v - \tau_v) \cdot X_{uv}) \right) \right] \\
&= \mathbf{E} \left[\mathbf{E}_X \exp \left(\sum_{\langle u v \rangle \in E'} i\sqrt{\beta}(\sigma_u + \tau_u - \sigma_v - \tau_v) \cdot X_{uv} \right) \right],
\end{aligned}$$

where $X = (X_{uv})_{\langle u v \rangle \in E'}$ is a collection of i.i.d. standard Gaussian random vectors in \mathbb{R}^n . Everything being integrable, we are allowed to reverse the order of \mathbf{E} and \mathbf{E}_X (by Fubini's theorem). Upon doing so, we obtain

$$Z(\tau_0, \tau_1) = \mathbf{E}_X [\mathbf{E}(e^{Y_0(\sigma_0, \tau_0, X)} e^{Y_1(\sigma_1, \tau_1, X)})],$$

where

$$\begin{aligned}
Y_0(\sigma_0, \tau_0, X) &:= f_0(\sigma_0, \tau_0) + i\sqrt{\beta} \sum_{\langle u v \rangle \in E'} (\sigma_u + \tau_u) \cdot X_{uv} \\
Y_1(\sigma_1, \tau_1, X) &:= f_1(\sigma_1, \tau_1) - i\sqrt{\beta} \sum_{\langle u v \rangle \in E'} (\sigma_v + \tau_v) \cdot X_{uv}.
\end{aligned}$$

Notice that given X , the random variables Y_0 and Y_1 are independent. It follows that

$$Z(\tau_0, \tau_1) = \mathbf{E}_X [\mathbf{E}(e^{Y_0(\sigma_0, \tau_0, X)}) \mathbf{E}(e^{Y_1(\sigma_1, \tau_1, X)})] \quad (6.14)$$

$$\leq \sqrt{\mathbf{E}_X [|\mathbf{E} e^{Y_0(\sigma_0, \tau_0, X)}|^2] \mathbf{E}_X [|\mathbf{E} e^{Y_1(\sigma_1, \tau_1, X)}|^2]}, \quad (6.15)$$

where the inequality is an application Cauchy-Schwarz. We now calculate

$$\begin{aligned}
|\mathbf{E} e^{Y_0(\sigma_0, \tau_0, X)}|^2 &= \mathbf{E}(e^{Y_0(\sigma_0, \tau_0, X)}) \overline{\mathbf{E}(e^{Y_0(\sigma_0, \tau_0, X)})} = \mathbf{E}(e^{Y_0(\sigma_0, \tau_0, X)}) \mathbf{E}(\overline{e^{Y_0(\sigma_0, \tau_0, X)}}) \\
&= \mathbf{E}(e^{Y_0(\sigma_0, \tau_0, X)}) \mathbf{E}(\overline{e^{Y_0(\theta(\sigma_1), \tau_0, X)}}),
\end{aligned}$$

where the last equality follows from the fact that σ_0 has the same law as $\theta(\sigma_1)$. Upon noticing

$$f_0(\theta(\sigma_1), \tau_0) = f_1(\sigma_1, \theta(\tau_0)), \quad \tau_u = (\theta(\tau_0))_v \quad \text{for } \langle u v \rangle \in E',$$

we can write

$$\begin{aligned}
\mathbf{E}(\overline{e^{Y_0(\theta(\sigma_1), \tau_0, X)}}) &= \mathbf{E} \exp \left(f_0(\theta(\sigma_1), \tau_0) - i\sqrt{\beta} \sum_{\langle u v \rangle \in E'} (\sigma_u + \tau_u) \cdot X_{uv} \right) \\
&= \mathbf{E} \exp \left(f_1(\sigma_1, \theta(\tau_0)) - i\sqrt{\beta} \sum_{\langle u v \rangle \in E'} (\sigma_v + (\theta(\tau_0))_v) \cdot X_{uv} \right) = \mathbf{E}(e^{Y_1(\sigma_1, \theta(\tau_0), X)}).
\end{aligned}$$

Therefore,

$$\mathbf{E}_X [|\mathbf{E}(e^{Y_0(\sigma_0, \tau_0, X)})|^2] = \mathbf{E}_X [\mathbf{E}(e^{Y_0(\sigma_0, \tau_0, X)}) \mathbf{E}(e^{Y_1(\sigma_1, \theta(\tau_0), X)})] \stackrel{(6.14)}{=} Z(\tau_0, \theta(\tau_0)).$$

Similarly, we can obtain

$$\mathbf{E}_X [|\mathbf{E}(e^{Y_1(\sigma_1, \tau_1, X)})|^2] = Z(\theta(\tau_1), \tau_1),$$

and so (6.15) gives the desired inequality.

6.2 Proof of Lemma 2

Take any $\tau : V \rightarrow \mathbb{R}^n$. For $\alpha \in \mathbb{R}$, define the function

$$f(\alpha) := Z(\alpha\tau) = \mathbf{E}(e^{A(\alpha)}), \quad A(\alpha) = -\frac{\beta}{2} \sum_{\langle uv \rangle \in E} \|\sigma_u + \alpha\tau_u - \sigma_v - \alpha\tau_v\|^2.$$

By Lemma 1, f is maximum at the zero function. In particular,

$$f''(0) \leq 0. \tag{6.16}$$

To make use of this condition, we make the computation

$$f''(\alpha) = \mathbf{E}[(A''(\alpha) + [A'(\alpha)]^2)e^{A(\alpha)}],$$

where

$$\begin{aligned} A'(\alpha) &= -\beta \sum_{\langle uv \rangle \in E} (\sigma_u + \alpha\tau_u - \sigma_v - \alpha\tau_v) \cdot (\tau_u - \tau_v) \\ &= -\beta \sum_{\langle uv \rangle \in E} [(\sigma_u - \sigma_v) \cdot (\tau_u - \tau_v) + \alpha\|\tau_u - \tau_v\|^2] \\ \Rightarrow A''(\alpha) &= -\beta \sum_{\langle uv \rangle \in E} \|\tau_u - \tau_v\|^2. \end{aligned}$$

We now have

$$f''(0) = \mathbf{E}\left[\left\{-\beta \sum_{\langle uv \rangle \in E} \|\tau_u - \tau_v\|^2 + \left(\beta \sum_{\langle uv \rangle \in E} (\sigma_u - \sigma_v) \cdot (\tau_u - \tau_v)\right)^2\right\}e^{A(0)}\right].$$

Observe that

$$e^{A(0)} = \exp\left(-\frac{\beta}{2} \sum_{\langle uv \rangle \in E} \|\sigma_u - \sigma_v\|^2\right) = \exp\left(-\beta \sum_{\langle uv \rangle \in E} (1 - \sigma_u \cdot \sigma_v)\right) = e^{-\beta|E|} e^{\beta H(\sigma)},$$

which means we can express $f''(0)$ in terms of an expectation with respect to the Gibbs measure:

$$f''(0) = e^{-\beta|E|} \mathbf{E}(e^{\beta H(\sigma)}) \left\langle -\beta \sum_{\langle uv \rangle \in E} \|\tau_u - \tau_v\|^2 + \left(\beta \sum_{\langle uv \rangle \in E} (\sigma_u - \sigma_v) \cdot (\tau_u - \tau_v)\right)^2 \right\rangle.$$

Now (6.16) shows

$$\left\langle \left(\beta \sum_{\langle uv \rangle \in E} (\sigma_u - \sigma_v) \cdot (\tau_u - \tau_v)\right)^2 \right\rangle \leq \beta \sum_{\langle uv \rangle \in E} \|\tau_u - \tau_v\|^2. \tag{6.17}$$

More generally, if $\tau : V \rightarrow \mathbb{C}^n$, we can apply (6.17) to $\operatorname{Re} \tau$ and $\operatorname{Im} \tau$ separately and then obtain

$$\left\langle \left| \beta \sum_{\langle uv \rangle \in E} (\sigma_u - \sigma_v) \cdot (\tau_u - \tau_v) \right|^2 \right\rangle \leq \beta \sum_{\langle uv \rangle \in E} \|\tau_u - \tau_v\|^2, \tag{6.18}$$

where now $x \cdot y$ denotes the standard inner product in \mathbb{C}^n :

$$x \cdot y = \sum_{i=1}^n x_i \overline{y_i}.$$

Next we make some simplifying calculations. First,

$$\begin{aligned}
\sum_{\langle u v \rangle \in E} (\sigma_u - \sigma_v) \cdot (\tau_u - \tau_v) &= \sum_{\langle u v \rangle \in E} \sigma_u \cdot (\tau_u - \tau_v) - \sum_{\langle u v \rangle \in E} \sigma_v \cdot (\tau_u - \tau_v) \\
&= \frac{1}{2} \sum_{u \in V} \sum_{v \in \mathcal{N}(u)} \sigma_u \cdot (\tau_u - \tau_v) - \frac{1}{2} \sum_{v \in V} \sum_{u \in \mathcal{N}(v)} \sigma_v \cdot (\tau_u - \tau_v) \\
&= -\frac{1}{2} \sum_{u \in V} \sigma_u \cdot (\Delta\tau)_u - \frac{1}{2} \sum_{v \in V} \sigma_v \cdot (\Delta\tau)_v \\
&= -\sum_{u \in V} \sigma_u \cdot (\Delta\tau)_u,
\end{aligned} \tag{6.19}$$

where

$$(\Delta\tau)_u := \sum_{v \in \mathcal{N}(u)} \tau_v - \tau_u.$$

Replacing $\sigma_u - \sigma_v$ in the above computation by $\tau_u - \tau_v$, we similarly obtain

$$\sum_{\langle u v \rangle \in E} \|\tau_u - \tau_v\|^2 = -\sum_{u \in V} \tau_u \cdot (\Delta\tau)_u.$$

Let $\mathbf{1} \in \mathbb{R}^n$ be the vector of all 1's, and make the specific choice of τ given by

$$\tau_u = e^{ik \cdot u} \mathbf{1}, \quad k \in V^* \setminus \{0\}.$$

In this case,

$$\begin{aligned}
(\Delta\tau)_u &= \sum_{v \in \mathcal{N}(u)} (e^{ik \cdot v} - e^{ik \cdot u}) \mathbf{1} = \sum_{j=1}^d [e^{ik \cdot (u+e_j)} + e^{ik \cdot (u-e_j)} - 2e^{ik \cdot u}] \mathbf{1} \\
&= \left(\sum_{j=1}^d 2(\cos(k_j) - 1) e^{ik \cdot u} \right) \mathbf{1}.
\end{aligned}$$

Therefore,

$$\sum_{\langle u v \rangle \in E} \|\tau_u - \tau_v\|^2 = -\sum_{u \in V} \tau_u \cdot (\Delta\tau)_u = -\sum_{u \in V} 2n \sum_{j=1}^d (\cos(k_j) - 1) = n|V| \sum_{j=1}^d 2(1 - \cos k_j). \tag{6.20}$$

Also,

$$\left\langle \left| \sum_{u \in V} \sigma_u \cdot (\Delta\tau)_u \right|^2 \right\rangle = \left(\sum_{j=1}^d 2(1 - \cos k_j) \right)^2 \left\langle \left| \sum_{u \in V} (\sigma_u \cdot \mathbf{1}) e^{ik \cdot u} \right|^2 \right\rangle, \tag{6.21}$$

where

$$\begin{aligned}
\left\langle \left| \sum_{u \in V} (\sigma_u \cdot \mathbf{1}) e^{ik \cdot u} \right|^2 \right\rangle &= \left\langle \left(\sum_{u \in V} (\sigma_u \cdot \mathbf{1}) e^{ik \cdot u} \right) \overline{\left(\sum_{v \in V} (\sigma_v \cdot \mathbf{1}) e^{ik \cdot v} \right)} \right\rangle \\
&= \sum_{u, v \in V} e^{ik \cdot (u-v)} \langle (\sigma_u \cdot \mathbf{1}) (\sigma_v \cdot \mathbf{1}) \rangle.
\end{aligned}$$

The proof will be complete once we show

$$\langle (\sigma_u \cdot \mathbf{1})(\sigma_v \cdot \mathbf{1}) \rangle = \langle \sigma_u \cdot \sigma_v \rangle, \quad (6.22)$$

since then

$$\begin{aligned} \frac{1}{|V|} \sum_{u,v \in V} e^{ik \cdot (v-u)} \langle \sigma_u \cdot \sigma_v \rangle &\stackrel{(6.22)}{=} \frac{1}{|V|} \sum_{u,v \in V} e^{ik \cdot (v-u)} \langle (\sigma_u \cdot \mathbf{1})(\sigma_v \cdot \mathbf{1}) \rangle \\ &\stackrel{(6.21)}{=} \frac{1}{|V|} \left\langle \left| \sum_{u \in V} \sigma_u \cdot (\Delta \tau)_u \right|^2 \right\rangle \left(\sum_{j=1}^d 2(1 - \cos k_j) \right)^{-2} \\ &\stackrel{(6.19)}{=} \frac{1}{|V|} \left\langle \left| \sum_{\langle uv \rangle \in E} (\sigma_u - \sigma_v) \cdot (\tau_u - \tau_v) \right|^2 \right\rangle \left(\sum_{j=1}^d 2(1 - \cos k_j) \right)^{-2} \\ &\stackrel{(6.18)}{\leq} \frac{1}{|V|} \left(\frac{1}{\beta} \sum_{\langle uv \rangle \in E} \|\tau_u - \tau_v\|^2 \right) \left(\sum_{j=1}^d 2(1 - \cos k_j) \right)^{-2} \\ &\stackrel{(6.20)}{=} \frac{n}{\beta} \left(\sum_{j=1}^d 2(1 - \cos k_j) \right)^{-1}, \end{aligned}$$

which is exactly the claim.

To verify (6.22), consider the following fact. For any $u \neq v$ in V and indices $i \neq j$, the pair of coordinates $((\sigma_u)_i, (\sigma_v)_j)$ has the same joint distribution as $((\sigma_u)_i, -(\sigma_v)_j)$ under the Gibbs measure. This is because

$$H(\sigma) = H(\sigma'),$$

where σ' is the spin configuration obtained from σ by flipping the j -th coordinate at every site:

$$(\sigma'_w)_k = \begin{cases} (\sigma_w)_k & k \neq j \\ -(\sigma_w)_j & k = j. \end{cases}$$

This symmetry implies $\langle (\sigma_u)_i (\sigma_v)_j \rangle = 0$ whenever $i \neq j$. Therefore,

$$\langle (\sigma_u \cdot \mathbf{1})(\sigma_v \cdot \mathbf{1}) \rangle = \left\langle \left(\sum_{i=1}^n (\sigma_u)_i \right) \left(\sum_{j=1}^n (\sigma_v)_j \right) \right\rangle = \sum_{i,j=1}^n \langle (\sigma_u)_i (\sigma_v)_j \rangle = \sum_{i=1}^n \langle (\sigma_u)_i (\sigma_v)_i \rangle = \langle \sigma_u \cdot \sigma_v \rangle.$$

6.3 Proof of Theorem 1

Take any $a \in \mathbb{Z}^n$, and consider the quantity

$$\sum_{k \in V^*} e^{ik \cdot a} = \sum_{t_1, \dots, t_d = -L+1}^L \prod_{j=1}^d e^{i\pi t_j a_j / L} = \prod_{j=1}^d \sum_{t=-L+1}^L e^{i\pi t a_j / L},$$

which one may recognize as the d -dimensional discrete Fourier transform of the constant 1 function, evaluated at the frequency vector a . It is not surprising, then, that when $a_j \neq 0$, the product is 0. Indeed, by a geometric series computation, $a_j \in \mathbb{Z} \setminus \{0\}$ implies

$$\sum_{t=-L+1}^L e^{i\pi t a_j / L} = \frac{e^{i\pi a_j (-L+1)/L} (1 - e^{i2\pi a_j})}{1 - e^{i\pi a_j / L}} = 0.$$

Of course, if $a_j = 0$ for every j , then

$$\sum_{k \in V^*} e^{ik \cdot a} = |V^*| = |V|.$$

This discussion shows

$$\sum_{k \in V^*} e^{ik \cdot (v-u)} \langle \sigma_u \cdot \sigma_v \rangle = \begin{cases} |V| & u = v \\ 0 & u \neq v. \end{cases}$$

Using Lemma 2, we thus have

$$\begin{aligned} |V| &= \frac{1}{|V|} \sum_{u,v \in V} \sum_{k \in V^*} e^{ik \cdot (v-u)} \langle \sigma_u - \sigma_v \rangle \\ &= \frac{1}{|V|} \sum_{u,v \in V} \langle \sigma_u - \sigma_v \rangle + \frac{1}{|V|} \sum_{k \in V^* \setminus \{0\}} \sum_{u,v \in V} e^{ik \cdot (v-u)} \langle \sigma_u \cdot \sigma_v \rangle \\ &\leq \frac{1}{|V|} \sum_{u,v \in V} \langle \sigma_u - \sigma_v \rangle + \frac{n}{2\beta} \sum_{k \in V^* \setminus \{0\}} \frac{1}{\sum_{j=1}^d (1 - \cos k_j)}, \end{aligned}$$

which upon rearrangement reads

$$\frac{1}{|V|^2} \sum_{u,v \in V} \langle \sigma_u \cdot \sigma_v \rangle \geq 1 - \frac{n}{2\beta|V|} \sum_{k \in V^* \setminus \{0\}} \frac{1}{\sum_{j=1}^d (1 - \cos k_j)}. \quad (6.23)$$

It is now a standard observation of Riemann integration that

$$\lim_{L \rightarrow \infty} \frac{1}{|V|} \sum_{k \in V^* \setminus \{0\}} \frac{1}{\sum_{j=1}^d (1 - \cos k_j)} = \frac{1}{(2\pi)^d} \int_{(-\pi, \pi]^d} \frac{1}{\sum_{j=1}^d (1 - \cos x_j)} dx,$$

which is finite if $d \geq 3$. To why this is the case, recall that $1 - \cos x \approx x^2/2$ for x close to 0. Of course,

$$\sum_{j=1}^d x_j^2 = |x|^2,$$

and so we are really observing that

$$\begin{aligned} \int_{B(0,1)} \frac{1}{|x|^2} &= \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \int_0^1 \frac{1}{r^2} r^{d-1} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \cdots \sin(\phi_{d-2}) dr d\phi_1 \cdots d\phi_{d-2} d\phi_{d-1} \\ &\begin{cases} = \infty & d = 1, 2 \\ < \infty & d \geq 3. \end{cases} \end{aligned}$$

The result now follows from (6.23) by taking β and L sufficiently large.

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Disclaimer: These notes may contain factual and/or typographic errors.

0.4 Correlation inequalities

0.4.1 FKG inequality

First, we discuss the FKG inequality.

Let $S \subseteq \mathbb{R}$ be a finite or countable subset of \mathbb{R} , and let ρ be a probability mass function that is strictly positive.

First, we introduce some definitions:

Definition 3 For $x, y \in S^n$, $x \leq y$ if and only if $x_i \leq y_i$, $\forall i$.

Definition 4 The probability mass is said to satisfy the FKG lattice condition if

$$\forall x, y \in S^n, \rho(x)\rho(y) \leq \rho(x \wedge y)\rho(x \vee y)$$

Note that a probability measure will always satisfy the lattice condition.

Definition 5 A function $f : S^n \rightarrow \mathbb{R}$ is monotone increasing if

$$x \leq y \Rightarrow f(x) \leq f(y)$$

Theorem 6 Consider ρ that is strictly positive and satisfies lattice condition. Let X be distributed as ρ . Let f, g be two monotone increasing functions s.t.

$$\mathbb{E}f(X)^2 < \infty, \mathbb{E}g(X)^2 < \infty.$$

Then,

$$\text{Cov}(f(X), g(X)) \geq 0$$

Proof: By induction on n

For $n=1$, let X, Y be i.i.d. ρ distributed.

Then,

$$\mathbb{E}[(f(X) - f(Y))(g(X) - g(Y))] = 2\text{Cov}(f(X), g(X))$$

Since $n=1$, $X \geq Y$, or $X \leq Y$, and in either case, as f, g are monotone increasing,

$$(f(X) - f(Y))(g(X) - g(Y)) \geq 0.$$

Therefore,

$$\mathbb{E}[(f(X) - f(Y))(g(X) - g(Y))] \geq 0$$

Now, suppose that $n > 1$ and the inequality holds in all small dimensions.

Define

$$f_1(a) = \mathbb{E}(f(X)|X_1 = a)$$

We have an easy identity

$$\text{Cov}(A, B) = \mathbb{E}[\text{Cov}(A, B|C)] + \text{Cov}(E(A|C), E(B|C)),$$

Hence, we have

$$\text{Cov}(f(X), g(X)) = \mathbb{E}[f(X)g(X) - f_1(x_1)g_1(x_1)] + \mathbb{E}[f_1(x_1)g_1(x_1)] - \mathbb{E}[f(X)]\mathbb{E}[g(X)]$$

Let $X' = (X_2, \dots, X_n)$, then the conditional probability mass for X' given $X_1 = a$ is proportional to $\rho(a, x_2, \dots, x_n)$. Clearly, this also satisfies the lattice condition, and by the induction hypothesis, we have

$$\text{Cov}(f(X), g(X)|X_1 = a) \geq 0, \quad \forall a$$

Next, we want to show that f_1, g_1 are monotone increasing functions, and this will complete the proof by the $n = 1$ case

$$f_1(b) - f_1(a) = \frac{\sum_{x' \in \mathbb{S}^{n-1}} f(b, x')\rho(b, x')}{\sum_{x' \in \mathbb{S}^{n-1}} \rho(b, x')} - f_1(a)$$

Let $\tau(x') = \frac{\rho(b, x')}{\rho(a, x')}$

$$\begin{aligned} f_1(b) - f_1(a) &= \frac{\sum_{x' \in S^{n-1}} f(b, x')\tau(x')\rho(a, x')}{\sum_{x' \in S^{n-1}} \tau(x')\rho(a, x')} - f_1(a) \\ &\geq \frac{\sum_{x \in S^{n-1}} \rho(a, x)\tau(x)f(a, x)}{\sum_{x \in S^{n-1}} \tau(x)\rho(a, x)} - f_1(a) \\ &= \frac{\sum_{x \in S^{n-1}} f(a, x)\tau(x)f(a, x)}{\frac{\sum \rho(a, x)}{\sum \text{rho}(a, x)}} - f_1(a) \\ &= \frac{\mathbb{E}(f(X)\tau(X')|X_1 = a)}{E(\tau(X')|X_1 = a)} - \mathbb{E}(f(X)|x_1 = a) \\ &= \frac{\text{Cov}(f(a, X'), \tau(X')|X_1 = a)}{E(\tau(X')|X_1 = a)} \end{aligned}$$

We have that the distribution of X' given $X_1 = a$ satisfies the lattice condition and $f(a, \cdot)$ is monotone increasing function (m.i.f.).

Thus, to show that, we only need to show that τ is a m.i.f.

This is true as for $x', y' \in S^{n-1}$, $x' \leq y'$, ρ satisfies the lattice condition.

$$\rho(b, y')\rho(a, x') - \rho(a, y')\rho(b, x')$$

■

0.4.1.1 Application to Ising model

Let σ, σ' be two configuration, $\tau = \sigma \wedge \sigma'$, $\eta = \sigma \vee \sigma'$.

We want to show that

$$\sum_{\langle i, j \rangle} \sigma_i \sigma_j + \sum_{\langle i, j \rangle} \sigma'_i \sigma'_j \leq \sum_{\langle i, j \rangle} \tau_i \tau_j + \sum_{\langle i, j \rangle} \eta_i \eta_j$$

This is true as $\sigma_i \sigma_j + \sigma'_i \sigma'_j \leq \tau_i \tau_j + \eta_i \eta_j$, for each i, j , because of a simple fact that given a_1, \dots, a_n and b_1, \dots, b_n , $\sum_i a_i b_{\pi(1)}$ maximized when $b_{\pi(1)}, \dots, b_{\pi(n)}$ in the same order as a_1, \dots, a_n

0.4.2 Griffin's inequality

Let μ_1, \dots, μ_n be probability measure on \mathbb{R} , which are symmetric around 0.

For each subset $A \subset \{1, \dots, n\}$, let J_A be a non-neg real number

Define a prob measure γ on \mathbb{R}^n

$$d_\gamma(\eta) = \frac{1}{Z} \exp \left(\sum_A J_A \eta^A \right) \prod d\mu_i(\eta_i)$$

where $\eta^A = \prod_{i \in A} \eta_i$.

Theorem 7 Let j_δ denote expectation w.r.t. γ .

Then,

$$\forall A, \langle \eta_A \rangle \geq 0,$$

$$\forall A, B, \langle \eta^A \eta^B \rangle \geq \langle \eta^A \rangle \langle \eta^B \rangle$$

Now, we begin setting up notations and lemmas for proving the inequality.

For any i and any non-neg integer m

$$\int \xi_i^m d\mu_i(\xi_i) \geq 0 \tag{0.24}$$

Then

Given $\xi, \chi \in \mathbb{R}^n$, let

$$q = \xi - \chi$$

$$t = \xi + \chi$$

Lemma 8 $\forall i, \forall a, b$ non-neg integer

$$\int q_i^a t_i^b d\mu(\xi_i) d\mu(\chi_i) \geq 0$$

Proof: By the symmetry of μ_i , the integral remains the same if we replace $\xi_i \rightarrow -\chi_i, \chi_i \rightarrow -\xi_i$. Then, the integral becomes

$$\int (-q_i)^a t_i^b d\mu(\xi_i) d\mu(\chi_i)$$

Thus, if a odd, then the integral becomes 0. Similarly, if b is odd, then the integral is also 0. If a, b both even, then clearly the integral is nonnegative. ■

Now, we state another lemma.

Lemma 9 For any $A, \xi^A + \chi^A$ and $\xi^A - \chi^A$ are both polynomials in q, t with non-neg coefficients.

Proof:

$$\xi^A \pm \chi^A = 2^{-|A|} ((q+t)^A \pm (q-t)^A)$$

To show that this polynomial in (q, t) has non-neg coefficients, observe that a typical term in $(q+t)^A$ is has a corresponding term in $(q-t)^A$

The two terms either equal (when $|A-B|$ is even) and contribute to a 2 or cancel out in adding $\xi^A + \chi^A$, and the reverse happens when doing $\xi^A - \chi^A$.

Hence, all the coefficients are non-neg. ■ Now, let ξ, χ be independent copies from γ . Then $\forall A, B$

$$\langle \xi^A \xi^B \rangle - \langle \xi^A \rangle \langle \xi^B \rangle = \langle \xi^A (\xi^B - \chi^B) \rangle = \langle \text{a polynomial in } (q, t) \text{ with non-neg coefficients} \rangle$$

Now, if we can prove the following lemma, we will finish the proof for the inequality.

Lemma 10 Let P be the set of polynomials in (q, t) with non-neg coefficients.

Then, for all $f \in P, \langle f \rangle \geq 0$

Proof:

$$\begin{aligned} \langle f \rangle &= \frac{1}{Z^2} \int f \exp(\sum J_A) (\xi^A + \chi^A) d\mu(\xi) d\mu(\chi) \\ &= \frac{1}{Z^2} \sum_{k=0}^{\infty} \frac{1}{k!} \int f \left(\sum J_A (\xi^A + \chi^A) \right)^k \Pi d\mu_i(\xi_i) \Pi d\mu_i(\chi_i) \\ &\geq 0, \text{ by lemma 8} \end{aligned}$$

■

0.4.3 Examples

We'll give some examples of the application of these two inequalities.

0.4.3.1 FKG

Clearly, $\sigma \rightarrow \sigma_i$ is a monotone increasing function for all i

Then we have that $\langle \sigma_i \sigma_j \rangle \geq \langle \sigma_i \rangle \langle \sigma_j \rangle$ in the Ising model. In particular, for free or periodic boundary, $\langle \sigma_i \sigma_j \rangle \geq 0$

0.4.3.2 Application of Griffin's inequality

Consider the Ising model on $\Lambda \subset \mathbb{Z}^d$ with free bdy.

Suppose that we add one vertex v to Λ to get Λ' with prob density proportional to $\exp(\sum \sigma_i \sigma_j)$

Take any $A \subset \Lambda$ s.t. no member of A is a neighbour of v .

$$\frac{\partial}{\partial J} \langle \sigma^A \rangle = \langle \sigma^A \left(\sum_{j \in N(v)} \sigma_i \sigma_j \right) \rangle - \langle \sigma^A \rangle \langle \sum_{j \in N(v)} \sigma_i \sigma_j \rangle \geq 0$$

When $J = 0$, $\langle \sigma^A \rangle$ is the expectation of σ^A in the Ising model on Λ

$J = \beta$, $\langle \sigma^A \rangle$ is the expectation of σ^A in the Ising model on Λ' .

This means that the new expectation \geq old expectation wherever you added the new spin.

But $0 \leq \langle \sigma^A \rangle \leq 1$, so if we have a sequence Λ_n increase to \mathbb{Z}^d , $\langle \sigma^A \rangle$ must increase to a limit, and is this limit independent of the choice of the sequence.

Lastly, observe that expectation like $\langle \sigma^A \rangle$ completely determines the finite dim distribution of any prob measure on $\{-1, 1\}^{\mathbb{Z}^d}$ with the product σ -algebra

0.4.3.3 Ising model with an external field

By Griffin's inequality, this model has an infinite volume limit.

Definition 11 *Expected magnetization*

$$\mathbb{E} \left[\frac{1}{|\Lambda|} \right] = m(\beta, h, \Lambda)$$

We want to know What is $m(\beta, h) = \lim_{\Lambda \rightarrow \mathbb{Z}^d} m(\beta, h, \Lambda)$. If $h = 0$, then this is 0.

We will approach \mathbb{Z}^d through $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$

Let $m_n = m(\beta, h, \Lambda_n)$

Theorem 12 $\lim m_n$ exists for almost every h and is non-decreasing in h

Theorem 13 If β is large enough, then $m(\beta, h)$ has a jump discontinuous at 0.

Theorem 14 *Lee-yang theorem for magnetization*

On $(0, \infty)$ and $(-\infty, 0)$, $m(\beta, h)$ is analytic of h , for any β , and hence no other phase transition.

Let $Z_n(h) = Z(\beta, h, \Lambda_n)$, with β fixed, and $F_n(h) = \frac{1}{|\Lambda_n|} \log z_n(h)$ be the free energy.

Lemma 15 $\forall h \in R, F(h) = \lim F_n(h)$ exists

Simple calculation gives

$$F'_n(h) = E(\text{magnetization})F''_n(h) = |\Lambda_n| \text{Var}(\text{magnetization})$$

Hence F_n is convex and F is also convex of h

Fact a convex function is a.e. differentiable

Then the limit exists and equals $F'(h)$.

Lecture 9: May 1

Lecturer: Sourav Chatterjee

Scriber: Jun Yan

Disclaimer: *These notes may contain factual and/or typographic errors.*

Today we'll prove Claim 1 and Theorem 2 below. First let's recall where we were last time. Write $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$. Consider

$$Z_n = \sum_{\sigma \in \{-1, 1\}^n} \exp\left(\beta \sum \sigma_j \sigma_k + h \sum \sigma_j\right).$$

Fix β , then we can write Z_n as a function of $h : Z_n(h)$. Let $F_n = \frac{1}{|\Lambda_n|} \log Z_n$. We showed that $\forall h$, $\lim_{n \rightarrow \infty} F_n(h)$ exists.

Call the limit as $F(h)$. Then $\langle \text{magnetization} \rangle = F'(h)$. F_n is convex $\implies F'_n$ is increasing in h , $\implies \lim F'_n(h)$ exists for a.e. h (where F is differentiable). We'll prove the following claim.

Claim 16 F' has a jump discontinuity at 0 if β is large.

Proof: Take any $h > 0$. Let $h_n = \frac{\alpha}{|\Lambda_n|}$. By the monotonicity of F'_n , $F'_n(h) \geq F'_n(h_n)$ if n is large. Write $m = m(\sigma) = \frac{\sum \sigma_j}{|\Lambda_n|}$. Then

$$F'_n(h_n) = \langle m \rangle_{h_n} = \frac{\langle m e^{\alpha m} \rangle_0}{\langle e^{\alpha m} \rangle_0} = \frac{\langle m \rangle_0 + \alpha \langle m^2 \rangle_0 + O(\alpha^2)}{1 + \alpha \langle m \rangle_0 + O(\alpha^2)},$$

where $O(\alpha^2)$ is a quantity bounded by $C\alpha^2$ for a universal constant C . Notice that $\langle m \rangle_0 = 0$, thus if β is large

$$\langle m \rangle_{h_n} = \frac{\alpha \langle m^2 \rangle_0 + O(\alpha^2)}{1 + O(\alpha^2)} = \frac{1}{|\Lambda_n|^2} \sum \langle \sigma_j, \sigma_k \rangle \geq C(\beta) > 0$$

by taking a small enough α . Thus $F'(h) = \lim F'_n(h) \geq C(\beta) > 0$. Similarly, $F'(-h) \leq -C(\beta) < 0$, which implies the jump discontinuity. ■

Next we prove the following Lee-Yang Theorem.

Theorem 17 (Lee-Yang) *For any fixed β , F is analytic on $(0, \infty)$ and $(-\infty, 0)$. (meaning that F has convergent power series on the neighborhood of every points.)*

Proof: Fix β , and take any complex h . Define $Z_n(h)$ as before. Let $\Omega = \{z \in \mathbb{C} : \text{Re}(z) \geq |\text{Im}(z)|\}$. Lee and Yang proved that all zeros of Z_n lie on the imaginary axis. We'll prove that Z_n has no zeros in Ω , which is enough for our purpose.

Note the following fact: If f is an analytic function on a simply connected domain Ω and f has no zeros in Ω , then \exists analytic g on Ω s.t. $e^g = f$.

Thus, on our Ω , $F_n(h) = \frac{1}{|\Lambda_n|} \log Z_n$ is defined. Moreover, since Z_n is positive on $(0, \infty)$, $\log Z_n$ must have imaginary part = integer times 2π on $(0, \infty)$, \implies the integer must be same throughout. Thus we can choose a version of $\log Z_n$ such that it agrees with our usual $\log Z_n$ on $(0, \infty)$.

Consequence: F_n has an analytic extension to Ω . Let $G_n := |Z_n|^{\frac{1}{|\Lambda_n|}}$, then because

$$\begin{aligned} |Z_n| &= \left| \sum \exp(\beta \sum \sigma_j \sigma_k + h \sum \sigma_j) \right| \\ &\leq \sum \exp(\beta \sum \sigma_j \sigma_k + \operatorname{Re}(h) \sum \sigma_j) = Z_n(\operatorname{Re} h), \end{aligned}$$

we see that $|G_n(h)| \leq \exp(F_n(\operatorname{Re} h))$. Note that:

(1) G_n converges pointwisely on $(0, \infty)$.

(2) G_n is uniformly bounded on compact subsets of Ω (with respect to n).

By (1), (2) and the Vitali's convergence theorem for analytic function, we see that $G = \lim G_n$ exists on Ω and is analytic. We'll prove that:

(3) $\forall h \in \Omega$, $|Z_n| \geq Z_n(\operatorname{Re} h - |\operatorname{Im} h|)$.

If (3) is true, $\forall h \in \Omega$ we have $|G_n(h)| \geq G(\operatorname{Re} h - |\operatorname{Im} h|) > 0$, and thus G has no zeroes in Ω , \implies it has an analytic logarithm. Thus $\log G$ can be made to agree with F on $(0, \infty)$, $\implies F$ has a continuous extension to Ω .

Proof of (3): Fix n . Note that

$$|Z_n(h)|^2 = \sum_{\sigma, \tau} \exp\left(\beta \sum (\sigma_j \sigma_k + \tau_j \tau_k) + h \sum \sigma_j + \bar{h} \sum \tau_j\right).$$

Let $z_j = (\sigma_j + \tau_j)/2 + (\sigma_j - \tau_j)/2i$, then $|z_j| = 1$. After some algebra we have $\operatorname{Re}(z_j \bar{z}_k) = (\sigma_j \sigma_k + \tau_j \tau_k)/2$, and thus

$$\sigma_j \sigma_k + \tau_j \tau_k = z_j \bar{z}_k + z_k \bar{z}_j.$$

Similarly we can verify that

$$h \sigma_j + \bar{h} \tau_j = (\operatorname{Re} h + \operatorname{Im} h) z_j + (\operatorname{Re} h - \operatorname{Im} h) \bar{z}_j.$$

Hence

$$|Z_n(h)|^2 = \sum_{\sigma, \tau} \exp\left(\beta \sum (z_j \bar{z}_k + z_k \bar{z}_j) + \sum (\operatorname{Re} h + \operatorname{Im} h) z_j + \sum (\operatorname{Re} h - \operatorname{Im} h) \bar{z}_j\right).$$

If $h \in \Omega$, then the expression in the bracket is a polynomial in (z_j, \bar{z}_j) with non-negative coefficients. $\implies \exp(\dots)$ is a convergent power series in (z_j, \bar{z}_j) with non-negative coefficients. Moreover, the coefficients are monotone increasing function of $(\operatorname{Re} h + \operatorname{Im} h, \operatorname{Re} h - \operatorname{Im} h)$. It is easy to see that for any $n \in \mathbb{Z}$,

$$\sum_{\tau_j, \sigma_j \in \{-1, 1\}} z_j^n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 0, & \text{if } 2 \mid n, 4 \nmid n \\ 4, & \text{if } 4 \mid n \end{cases}.$$

Note that $|z_j| = 1$ implies that $\bar{z}_j = z_j^{-1}$, $\implies \forall n, m \in \mathbb{Z}$, $\sum_{\tau_j, \sigma_j \in \{-1, 1\}} z_j^m \bar{z}_j^n \geq 0$. Thus, $\sum_{\sigma, \tau}$ (any monomial in $(z_j, \bar{z}_j)_{j \in \Lambda_n}$) ≥ 0 . This implies that $|Z_n(h)|^2$ is a sum of non-negative terms, each of which is a non-negative monotone increasing function of $(\operatorname{Re} h + \operatorname{Im} h, \operatorname{Re} h - \operatorname{Im} h)$. Let $h' = \operatorname{Re} h - |\operatorname{Im} h|$. Then

$$(\operatorname{Re} h' + \operatorname{Im} h', \operatorname{Re} h' - \operatorname{Im} h') \leq (\operatorname{Re} h + \operatorname{Im} h, \operatorname{Re} h - \operatorname{Im} h),$$

and it implies that $|Z_n(h)|^2 \geq |Z_n(h')|^2 = Z_n(\operatorname{Re} h - |\operatorname{Im} h|)^2$. ■

Lecture 10: May 3

Lecturer: Sourav Chatterjee

Scriber: Jun Yan

Disclaimer: These notes may contain factual and/or typographic errors.

10.5 Lattice Gauge Theory

In this lecture we introduce the lattice gauge theory.

Motivation: First we do some comparisons between classical particle and field. We denote a path as $q(t)$, $0 \leq t \leq T$, $q(0) = a, q'(0) = v$. Let S be an action on the path q , such that

$$S(q) = \int_0^T \left(\frac{1}{2} q'(t)^2 - V(q(t)) \right) dt.$$

The minimizer of $S(q)$ gives Newton's equation: $q''(t) = -V'(q(t))$. This is the classical particle case. For field, we can do the same thing, that is, we can calculate the minimizer of some function acting on fields. Quantization in the particle case: consider the measure $\exp(-S(q))D(q)$, where $D(q)$ = "Lebesgue measure on path space". This is actually Wiener measure weighted by $\exp(-\int_0^T V(q(t))dt)$. We want to put measure on fields, too. However, the building of the measure is not developed well, so people took a look at the discrete one, which is the lattice gauge theory.

Yang Mills Theory: Components of the standard model of Quantum Mechanics, not mathematically well-defined.

Lattice Gauge Theory: Discrete version of the Yang Mills Theory, mathematically well defined.

Definition: Write d -dimension, G -compact Liegroup ($SU(N), SO(n), U(n), O(n)$). In physics, $SO(3)$ corresponds to strong force, and $SO(2)$ corresponds to weak force (no one knows why). Denote ρ as a representation of G , which means that ρ is a group homomorphism from G to $GL(V)$, where V is a complex vector space, and $GL(V)$ is the invertible linear maps on V . We further require ρ to be irreducible, meaning that V doesn't have non-trivial invariant subspace for ρ . Recall the following fact:

Fact: G is a compact Lie group $\implies V$ must be finite dimensional.

Write $\chi_\rho : \chi_\rho(g) = \text{Tr}(\rho(g))$ for $g \in G$, which is the character of ρ . Take $\Lambda \in \mathbb{Z}^d$, denote by $E(\Lambda)$ = edge set of Λ (undirected). Denote the space of configurations as $G^{E(\Lambda)}$. Each edge has a positively and a negatively oriented version. If $g = (g(e))_{e \in E(\Lambda)}$ is a configuration, then we follow the convention that $g_{xy} = g_{yx}^{-1} = g(e)$, if e is the edge $\langle x, y \rangle$ and xy is the positive orientation. We define the Plaquette p as a unit square, with its four edges $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_4$. It is always a two-dim object, even though the space might be in higher dimension. Write $g_p = g_{e_1} g_{e_2} g_{e_3} g_{e_4}$. The lattice gauge theory defines a probability measure on the space of configurations, with density proportional to $\exp(\beta \sum_p \text{Re}(\chi_\rho(g_p)))$, with respect to the Haar measure on G . The Yang Mills continuity problem is that: this measure has a continuous version after some scaling.

Write $\beta = \frac{1}{g_0^2}$, where g_0 = coupling strength. Then β small corresponds to strong coupling regime, β large corresponds to weak coupling regime.

YM mass gap problem (discrete version): Prove exp decay of correlation at all large β in $d = 4, G = SU(3), \rho$ = standard representation.

Wilson loop: Take any loop $l = e_1 e_2 \dots e_n$ in \mathbb{Z}^d , define the Wilson loop variable $w_l = \text{Re}(\chi(g(e_1) \dots g(e_n)))$. Here is the Wilson conjecture: In $d = 4$, G -non-abelian, we should have $|\langle w_l \rangle| \leq \exp(-c(\beta)\text{area}(l))$, where $\text{area}(l)$ = minimum surface area enclosed by l .

This is supposed to explain confinement of quarks. Since suppose that two quarks are at distance R of each other, let l be a rectangle with edge length R, T, R, T . Then Wilson gave a physics argument to show that the potential $V(R)$ between the quarks can be obtained as $\lim_{T \rightarrow \infty} -\frac{1}{T} \log \langle w_l \rangle$. So if the area law holds, then $V(R) \geq cR$, which confines quarks. The area law for small β for any dimensions and $G = SU(n)$ or $U(1)$, l -rectangles is proved. We'll do a probability proof of this one.

Remark: However this result is not useful for the general area law, since this method works for the abelian group, but the general one shouldn't hold for the abelian group.

Lecture 11/12: May 8 & 10

Lecturer: Sourav Chatterjee

Scriber: Mark Perlman

Disclaimer: *These notes may contain factual and/or typographic errors.*

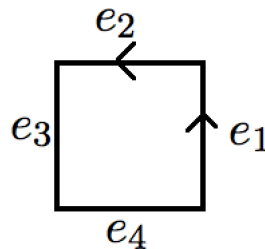
0.6 Lattice Gauge Theory

We're on \mathbb{Z}^d for $d \geq 2$. G is one of the following groups: $SU(n)$, $U(n)$, $O(n)$ or $SO(n)$ for $n \geq 1$, and ρ is the standard representation. We let $\Lambda \subset \mathbb{Z}^d$, and denote by $E(\Lambda)$ the set of positively oriented edges. Then the set of configurations is $G^{E(\Lambda)}$.

For a parameter β , p.m. has density with respect to Haar measure given by:

$$\frac{1}{2} \exp \left(\beta \sum_p \text{Re}(\text{Tr}(g_p)) \right),$$

where $g_p = g_{e_1}g_{e_2}g_{e_3}g_{e_4}$ for a square with edges e_1, e_2, e_3 and e_4 :



Denote by l a loop in \mathbb{Z}^d , that is, l is a sequence of edges $e_1e_2 \cdots e_m$ forming a closed path. Define $w_l = \text{Re}(\text{Tr}(g_{e_1} \cdots g_{e_m}))$.

Theorem 18 *If β is small enough, depending on d , then*

$$|\langle w_l \rangle| \leq e^{-c(\beta,d)\text{area}(l)}.$$

We will only prove this when l is a rectangle. Note that the bound is independent of Λ as long as Λ is large enough. This theorem tells us that the infinite volume limit exists by similar compactness arguments as in the Ising model.

It is conjectured that the infinite volume limit is unique for all β ; this has only been proved for small β .

Exercise 1 *Establish exponential decay of correlations at small β . Using this, prove the uniqueness of the infinite volume limit.*

Definition 19 *Take any infinite volume limit. The -quark-antiquark tension at separation R is defined as*

$$V(R) = \lim_{T \rightarrow \infty} -\frac{1}{T} \log \langle w_l \rangle$$

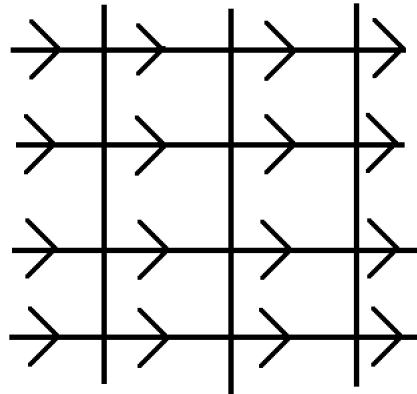
assuming the limit exists, where l is an $R \times T$ rectangle:



A model is said to exhibit quark confinement if $\lim_{R \rightarrow \infty} V(R) = \infty$.

Proposition 20 *If the area law holds, then so does quark confinement.*

Proof: $d = 2$, layers are independent after conditioning on the horizontal edges:



If $d = 3$, pick one of the 3 directions of the edges and condition on edges in the other two directions, to obtain $2d$ layers after conditioning. In general d -dimension, pick one cardinal direction, and condition on edges not in that direction. That is, condition on edges that are **not** of the form

$$((x_1, \dots, x_i, \dots, x_d), (x_1, \dots, x_i + 1, \dots, x_d))$$

for a single fixed i . After conditioning, the system consists of independent $(d - 1)$ -dimensional layers, and each layer is like a spin system in \mathbb{Z}^{d-1} in terms of neighbor interaction.

Fix any layer, and consider the edges as vertices in \mathbb{Z}^{d-1} . The conditional probability density of a configuration within a layer is proportional to

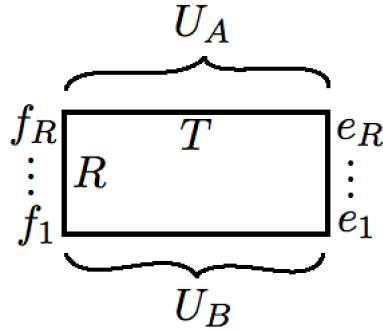
$$\exp \left(\beta \sum_{\langle ef \rangle} \text{Re}(\text{Tr}(g_e A_{ef} g_f B_{ef})) \right),$$

where the matrices A_{ef} and B_{ef} are deterministic and defined through the conditioning. This density has $g \rightarrow -g$ symmetry, and therefore the conditional expectation of g_e is zero for every edge.

Moreover, by a coupling argument as before, we get exponential decay of correlations at small β in this conditional law. The exponent does not depend on the conditioning, because for small β , the law is uniformly close to Haar measure. A and B are elements of a compact group and are therefore bounded, so if e, f are edges at distance 2^m , then for any two elements g_e^{ij} and g_f^{kl} ,

$$|\langle g_e^{ij} g_f^{kl} \rangle_{\text{cond}}| \leq e^{-C(\beta-d)m},$$

where here $\langle \cdot \rangle_{\text{cond}}$ denotes the conditional expectation. Denoting the edges of the rectangle l as



we have

$$w_l = \text{Re}(\text{Tr}(U_A g_{e_1} \dots g_{e_R} U_B g_{f_R}^* \dots g_{f_1}^*)).$$

We will show that, for any conditioning:

$$|\langle w_l \rangle_{\text{cond}}| \leq e^{-CTR},$$

again, the above denoting conditional expectation. This implies the area law. We the trace inside w_l as

$$\text{Tr}(\dots) = \sum_{i_1, \dots, i_{2R+2}} U_A^{i_1 i_2} g_{e_1}^{i_2 i_3} \dots g_{e_R}^{i_{R+1} i_{R+2}} U_B^{i_{R+2} i_{R+3}} g_{f_R}^{-i_{R+4} i_{R+3}} \dots g_{f_1}^{-i_1 i_{2R+2}}.$$

Then taking conditional expectation,

$$\langle \text{Tr}(\dots) \rangle_{\text{cond}} = \sum U_A^{i_1 i_2} U_B^{i_{R+2} i_{R+3}} \langle g_{e_1}^{i_2 i_3} g_{f_1}^{-i_1 i_{2R+2}} \rangle_{\text{cond}} \dots \langle \dots \rangle_{\text{cond}}.$$

And therefore each summand is bounded by $(e^{-C(\beta)T})^R$ in absolute value. The number of summands is bounded by e^{CR} , so if we choose β small enough that $C(\beta) > C$, we obtain the desired inequality. ■

0.7 Gaussian Free Field

Definition 21 Let $G = (V, E)$ be a finite undirected graph with no self groups, which we'll assume is connected. The ρ -Gaussian free field on G is a Gaussian field $(\phi_v)_{v \in V}$ with density proportional to

$$\rho = \exp \left(-\frac{1}{2} \sum_{(u,v) \in E} (\phi_u - \phi_v)^2 \right)$$

This definition immediately runs into the problem that $\int \rho = \infty$ because of translation invariance. There are two ways to solve this problem:

1. Designate a subset $\partial V \subset V$ as the “boundary” of V and fix the value of ϕ on ∂V (e.g. zero). If $\partial V = \emptyset$, the density is integrable.
2. Put a “mass” term in the Hamiltonian:

$$\exp \left(-\frac{1}{2} \sum_{(u,v) \in E} (\phi_u - \phi_v)^2 - \frac{m^2}{2} \sum_u \phi_u^2 \right).$$

This is referred to as a m -massive free field.

Example 1 Let $V = \{0, \dots, n-1\}^d$ with the natural boundary $\partial V = \{x \in V : \text{some coordinate of } x \text{ is } 0 \text{ or } n-1\}$.

Example 2 Let $V = \{0, \dots, n-1\}$ with $\partial V = \{0\}$, $\phi_0 = 0$, then the Gaussian free field is a random walk with Gaussian increments.

Example 3 Let $V = \{0, \dots, n-1\}$ with $\partial V = \{0, n-1\}$ and a zero boundary condition. Then the Gaussian free field is a BB evaluated at $0, \dots, n-1$.

Remark 22 From here on we'll assume a zero boundary condition on ∂V .

Definition 23 Given a graph G , the discrete Laplacian of a function $f : G \rightarrow \mathbb{R}$ is given by

$$\Delta f(u) = \sum_{v \in N(u)} f(v) - f(u),$$

where $N(u)$ is the neighborhood of u .

Proposition 24 Δ is given by a strictly negative definite matrix.

Proof: We calculate

$$\begin{aligned} \sum_{\langle u,v \rangle} (f(u) - f(v))^2 &= \sum_{\langle u,v \rangle} f(u)(f(u) - f(v)) - \sum_{\langle u,v \rangle} f(v)(f(u) - f(v)) \\ &= \frac{1}{2} \sum_u \sum_{w \in N(u)} f(u)(f(u) - f(w)) - \frac{1}{2} \sum_v \sum_{w \in N(v)} f(v)(f(w) - f(v)) \\ &= -\frac{1}{2} \sum_u f(u) \Delta f(u) - \frac{1}{2} \sum_v f(v) \Delta f(v) \\ &= -\sum_u f(u) \Delta f(u). \end{aligned}$$

■

Corollary 25 The covariance matrix of the Gaussian free field is $(-\Delta)^{-1}$.

0.7.1 Random Walk Representation of Covariance

Discrete Green's functions are typically complicated objects, but become more manageable when viewed in the following random walk representation:

Attach to each edge of G a Poisson clock that rings at rate 1; that is, every edge has a clock that rings after a random time distributed as an exponential distribution with rate 1, and then starts another exponential clock. The clocks on each edge are assumed independent of each other. Pick a vertex $u \in \text{int}(V)$, where $\text{int}(V) = V \setminus \partial V$. Start a random walk at u which remains at u until an edge attached to u rings, at which point cross that edge. Continue doing this until the random walk hits ∂V , at which point the walk stops.

Recall that a stopping time τ is a random time such that the event $\{\tau \leq t\}$ depends only on what has happened up to time t . Recall the memoryless property of Poisson processes: for any stopping time τ , the process on $[\tau, \infty)$ is the same as another Poisson process with the same rate. Thus in the above random walk, when the process jumps across an edge (i.e. we pick τ to be the first time the process jumps across an edge), all the clocks "reset." That is to say, this random walk is a Markov process.

Theorem 26 Denote $G(u, v) = \text{Cov}(\phi_u, \phi_v) = \mathbb{E}(\phi_u \phi_v)$. For each $v \in V$,

$$G(u, v) = \mathbb{E}_u(\text{time spent at } v),$$

where \mathbb{E}_u denotes expectation conditional on the random walk starting at u .

Proof: Fix v . Define $f(u) = \mathbb{E}_u(\text{time spent at } v)$, and let $g(u) = \mathbb{E}(\phi_u \phi_v)$. We want to prove that $f = g$. We already know that $f = g = 0$ on ∂V .

Let $u \in \text{int}(V)$, and assume first that $u \neq v$. We condition on the first jump. The fact that the walk is equally likely to jump to any neighbor of u , combined with the Markov property, tell us that

$$f(u) = \frac{1}{|N(u)|} \sum_{w \in N(u)} f(w),$$

and therefore $\Delta f(u) = 0$.

Now assume $u = v$. Then we again condition on the first jump, but now once we jump away from v , we've spent some time at v . Thus,

$$f(v) = \frac{1}{|N(v)|} \sum_{w \in N(v)} f(w) + \mathbb{E}(\text{time until first jump from } v).$$

The time until the first jump is the minimum of $|N(v)|$ i.i.d. exponential random variables (the clocks on every edge touching v), which is equal in distribution to an exponential with parameter $|N(v)|$. Therefore the expected time until the first jump is $\frac{1}{|N(v)|}$:

$$f(v) = \frac{1}{|N(v)|} \sum_{w \in N(v)} f(w) + \frac{1}{|N(v)|}.$$

Therefore $\Delta f(v) = -1$.

Now we prove that g satisfies the same Laplacian equations. Assume first that $u \neq v$. From the density of the Gaussian free field, we see that given $(\phi_w)_{w \neq u}$, the density of ϕ_u is proportional to

$$\begin{aligned} \exp\left(-\frac{1}{2} \sum_{w \in N(u)} (\phi_u - \phi_w)^2\right) &= \exp\left(-\frac{1}{2}|N(u)|\phi_u^2 + \sum_{w \in N(u)} \phi_u \phi_w\right) \\ &= \exp\left(-\frac{|N(u)|}{2} \left(\phi_u - \frac{1}{|N(u)|} \sum_{w \in N(u)} \phi_w\right)^2\right). \end{aligned}$$

Therefore, the conditional law of ϕ_u is normal with parameters

$$N\left(\frac{1}{|N(u)|} \sum_{w \in N(u)} \phi_w, \frac{1}{|N(u)|}\right),$$

and thus we calculate

$$\begin{aligned}
 g(u) &= \mathbb{E}(\phi_u \phi_v) \\
 &= \mathbb{E}[\mathbb{E}(\phi_u \mid (\phi_w)_{w \neq u}) \phi_v] \\
 &= \mathbb{E} \left[\frac{1}{|N(u)|} \sum_{w \in N(u)} \phi_w \phi_v \right] \\
 &= \frac{1}{|N(u)|} \sum_{w \in N(u)} g(w).
 \end{aligned}$$

Therefore $\Delta g(u) = 0$. On the other hand, when $u = v$,

$$\begin{aligned}
 g(v) &= \mathbb{E}(\phi_v^2) \\
 &= \mathbb{E}[\mathbb{E}(\phi_v^2 \mid (\phi_w)_{w \neq v})] \\
 &= \mathbb{E}[\text{Var}(\phi_v \mid (\phi_w)_{w \neq v})] + \mathbb{E}[\mathbb{E}(\phi_v \mid (\phi_w)_{w \neq v})^2] \\
 &= \frac{1}{|N(v)|} + \mathbb{E}[\phi_v \mathbb{E}(\phi_v \mid (\phi_w)_{w \neq v})] \\
 &= \frac{1}{|N(v)|} + \mathbb{E} \left[\phi_v \frac{1}{|N(v)|} \sum_{w \in N(v)} \phi_w \right] \\
 &= \frac{1}{|N(v)|} + \frac{1}{|N(v)|} \sum_{w \in N(v)} g(w).
 \end{aligned}$$

Therefore $\Delta g(v) = -1$.

If we let $h = f - g$, then $\Delta h = 0$ on $\text{int}(V)$, and $h = 0$ on ∂V . Since Δ is strictly negative definite, this implies that $h = 0$, i.e. $f = g$. \blacksquare

Corollary 27 *Since $G(u, v)$ is symmetric in u and v , we have $\mathbb{E}_u(\text{time spent at } v) = \mathbb{E}_v(\text{time spent at } u)$.*

Example 4 *In $d = 2$, let $V = \{0, \dots, n-1\}^2$ with the natural boundary and zero boundary condition. Consider a typical interior vertex u . Then*

$$\text{Var}(\phi_u) = G(u, u) = \mathbb{E}_u(\text{time spent at } u \text{ before hitting the boundary}).$$

Up to a constant, the random walk described above moves approximately as fast as a simple random walk. If X_t is a simple random walk on \mathbb{Z}^2 , then $\mathbb{P}_u(X_t = u) \sim \frac{\text{constant}}{t+1}$. So if T is some fixed large time, then

$$\mathbb{E}_u(\text{time spent at } u \text{ before } T) = \int_0^T \mathbb{P}_u(X_t = u) dt \sim C \log T.$$

Let τ be the time to hit the boundary, so τ is approximately the time for a simple random walk to attain a maximum or minimum of $n/2$, which takes about n^2 steps.

Exercise 2 *Show that $G(u, v) \sim C \log n$.*

Exercise 3 *We have $\mathbb{E}(\phi_u - \phi_v)^2 = G(u, u) + G(v, v) - 2G(u, v)$. Using the random walk representation, show that this quantity is $\sim C \log(|u - v| + 1)$ when $|u - v| \ll n$.*

0.7.2 Continuum Limit of Gaussian Free Field

Take $V = \{0, \dots, n-1\}^2$, scaled by $\frac{1}{n}$ so $V \subset [0, 1]^2$. Define a Gaussian free field as before on the scaled grid, where we don't scale the field (as compared to the $d = 1$ case, where we scaled the field along with the grid). Extend this to a field on $[0, 1]^2$ in some sensible way. Call this field ϕ , still depending on n . Let f be a smooth function with support in $(0, 1)^2$. Then

$$\int f \phi \approx \frac{1}{n^2} \sum_{u \in \text{grid}} f(u) \phi_u$$

is a normal random variable with mean 0 and variance

$$\sim \frac{1}{n^4} \sum_{u, v} f(u) f(v) G_n(u, v),$$

where G_n is the Green's function of the Gaussian free field on the scaled grid (i.e. the covariance matrix).

Fact 28 *If u_n, v_n are two sequences of grid points approaching two distinct points $x, y \in (0, 1)^2$, then*

$$G_n(u_n, v_n) \rightarrow G(x, y),$$

where $G(x, y)$ is the inverse of the Dirichlet Laplacian on $[0, 1]^2$, and is $\sim C \log |x - y|$ for $|x - y| \ll 1$.

Remark 29 *The continuum Gaussian free field is a random distribution, not a function. However, we can still see the covariance matrix; the only problem is that it's infinite on the diagonal.*

Lecture 13: May 15

Lecturer: Sourav Chatterjee

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Disclaimer: *These notes may contain factual and/or typographic errors.*

Gaussian Free Field with zero boundary condition

Let $G = (V, E)$, $\partial V \subseteq V$ be connected undirected graph. Probability density is given by

$$\exp\left(-\frac{1}{2} \sum_{\langle u, v \rangle} (\phi_u - \phi_v)^2\right),$$

where $\phi_u = 0$ for $u \in \partial V$. Connected graph implies that it is integrable and gives a probability measure.

Random walk representation is given by:

$$\langle \phi_u, \phi_v \rangle = E_u(\text{time spent at } v \text{ before hitting } \partial V)$$

Massive Free Field

Another way to get a probability measure is a massive free field:

$$\exp\left(-\frac{1}{2} \sum_{\langle u, v \rangle} (\phi_u - \phi_v)^2 - \frac{m^2}{2} \sum_u \phi_u^2\right).$$

Its random walk representation is:

$$\langle \phi_u, \phi_v \rangle = E_u(\text{time spent at } v)$$

but it dies after $\exp(m^2)$ random time.

Exercise: prove this.

Question from class: why is the mass “ m^2 ”? Answer: This comes from the potential of simple harmonic oscillator in QFT.

Continue limits in 2D

Free field is the simplest Euclidean QFT, rigorously defined. Formally, it gives a probability measure on (generalized) functions on \mathbb{R}^2 with probability density proportional to

$$\exp\left(-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi|^2-\frac{m^2}{2}\int_{\mathbb{R}^2}\phi_u^2\right)$$

with respect to a ‘‘Lebesgue measure’’ on function space.

Start with GFF on $\epsilon\mathbb{Z}^2$, which is a correct discretization:

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi|^2-\frac{m^2}{2}\int_{\mathbb{R}^2}\phi_u^2 &\rightarrow \frac{1}{2}\epsilon\sum_{\langle u,v\rangle}\left(\frac{\phi_u-\phi_v}{\epsilon}\right)^2+\frac{m^2\epsilon^2}{2}\sum_u\phi_u^2 \\ &= \frac{1}{2}\epsilon\sum_{\langle u,v\rangle\text{in}\epsilon\mathbb{Z}^2}(\phi_u-\phi_v)^2+\frac{m^2\epsilon^2}{2}\sum_u\phi_u^2.\end{aligned}$$

Define this for finite boxes in $\epsilon\mathbb{Z}^2$, and then take the infinite volume limit.

Why do we have uniqueness of the limit? You can see this by the random walk representation: random walk starting at point u will (with high probability) not ‘‘see’’ the boundary of the box before it dies, if the box is very big.

Therefore, $\langle\phi_u,\phi_v\rangle$ tends to a limit as boxes in $\epsilon\mathbb{Z}^2$.

Since Gaussian distributions are determined by covariances, the infinite volume limit is well-defined.

Now take $u,v\in\mathbb{R}^2$, $u\neq v$, and let u_ϵ,v_ϵ be the closest points to u,v in $\epsilon\mathbb{Z}^2$. Let $C_\epsilon(u_\epsilon,v_\epsilon)=\langle\phi_{u_\epsilon},\phi_{v_\epsilon}\rangle$ in $\epsilon\mathbb{Z}^2$. Then because transition kernel

$$p_t^\epsilon(x,y)=p_x(\text{random walk at }y\text{ at time }t),$$

$$\begin{aligned}C_\epsilon(u_\epsilon,v_\epsilon) &= \int_0^\infty p_t^\epsilon(u_\epsilon,v_\epsilon)P(t<\tau_\epsilon)dt \\ &= \int_0^\infty p_t^\epsilon(u_\epsilon,v_\epsilon)e^{-m^2\epsilon^2t}dt \\ &= \int_0^\infty \frac{1}{\epsilon^2}p_{s/\epsilon^2}^\epsilon(u_\epsilon,v_\epsilon)e^{-m^2s}ds \quad (\text{change of variable: }s=\epsilon^2t)\end{aligned}$$

As $\epsilon\rightarrow 0$, $p_t(u,v)=\frac{1}{4\pi t}e^{-\frac{|u-v|^2}{4t}}$, so

$$C_\epsilon(u_\epsilon,v_\epsilon)\rightarrow C(u,v)=\int_0^\infty \frac{1}{4\pi s}e^{|u-v|^2/4s}e^{-m^2s}ds.$$

Let ϕ^ϵ be a field on $\epsilon\mathbb{Z}^2$, extended to \mathbb{R}^2 in some reasonable way. Let f be a smooth function with exponential decay at infinity in \mathbb{R}^2 .

$$\int f\phi^\epsilon\sim\epsilon^2\sum_{u\in\epsilon\mathbb{Z}^2}f(u)\phi_u^\epsilon\sim N\left(0,\epsilon^4\sum_{u\in\epsilon\mathbb{Z}^2}f(u)f(v)C^\epsilon(u,v)\right)\xrightarrow{\epsilon\rightarrow 0}\int_{\mathbb{R}}\int_{\mathbb{R}}f(x)f(y)C(x,y)dx dy$$

The integral is finite because

$$C(x, y) \sim -\frac{1}{2\pi} \log(m|x - y|)$$

when $|x - y| \rightarrow 0$.

Minlos's theorem

Let $\mathcal{S}(\mathbb{R}^2)$ be the space of Schwartz functions, i.e. smooth functions f such that f and any of its higher derivatives decay faster than any polynomial at infinity. Namely, let \mathbb{Z}_+ be non-negative integers, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, define multi-index $|\alpha| = \alpha_1 + \alpha_2$, with $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, $x = (x_1, x_2) \in \mathbb{R}^2$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$, $p_{\alpha, \beta}(f) = \sup_{x \in \mathbb{R}^2} |x^\alpha D^\beta f(x)|$.

Define a metric on $C^\infty(\mathbb{R}^2)$,

$$d(f, g) = \sum_{\alpha, \beta} 2^{-|\alpha| - |\beta|} \frac{p_{\alpha, \beta}(f - g)}{1 - p_{\alpha, \beta}(f - g)},$$

then $\mathcal{S}(\mathbb{R}^2)$ is the complete metric space under this metric. It is a topological vector space.

$\mathcal{S}'(\mathbb{R}^2)$ = the continuous dual of $\mathcal{S}(\mathbb{R}^2)$, the space of tempered distributions.

$\mathcal{S}(\mathbb{R}^2) \hookrightarrow \mathcal{S}'(\mathbb{R}^2)$ is given by $\phi \mapsto \langle \phi, f \rangle$ on $\mathcal{S}'(\mathbb{R}^2)$, where $f \in \mathcal{S}(\mathbb{R}^2)$.

The σ -algebra generated by these maps is called the cylinder σ -algebra on $\mathcal{S}'(\mathbb{R}^2)$.

Given any probability measure on the cylinder σ -algebra, define its "character function"

$$S(f) = \int e^{i\langle \phi, f \rangle} d\nu(\phi).$$

Theorem 30 (Minlos's theorem) : function $S : \mathcal{S}'(\mathbb{R}^2) \rightarrow \mathbb{C}$ is a character for a probability measure on the σ -algebra of $\mathcal{S}'(\mathbb{R}^2)$ iff $S(0) = 1$, positive semi-definite, and S is continuous.

Proof: [Forward direction only] Let S_ϵ be a character function of ϕ^ϵ . $\forall f, \langle \phi, f \rangle$ converges in distribution

$\Rightarrow E(e^{i\langle \phi, f \rangle})$ converges to a limit

$\Rightarrow S(f) = \lim_{\epsilon \rightarrow 0} S_\epsilon(f)$ exists for all f .

then $S(0) = 1$, S is positive semi-definite, because each S_ϵ is.

$$\begin{aligned} |S_\epsilon(f) - S_\epsilon(g)| &= |E(e^{i\langle \phi^\epsilon, f \rangle} - e^{i\langle \phi^\epsilon, g \rangle})| \\ &\leq E(|\langle \phi^\epsilon, f - g \rangle|) \\ &\leq \sqrt{\text{Var}(\langle \phi^\epsilon, f - g \rangle)} \\ &\xrightarrow{\epsilon \rightarrow 0} \sqrt{\iint (f(x) - g(x))(f(y) - g(y)) C(x, y) dx dy} \end{aligned}$$

$\Rightarrow \{S_\epsilon\}_{\epsilon > 0}$ is uniformly equicontinuous, so by Arzelà–Ascoli theorem, S is continuous. ■

Physics Motivation

Quantum mechanics for single particle in 1D

let q be location, t be time.

A state is $\psi(q, t) \in L^2(\mathbb{R})$ with $\|\psi\|_{L^2} = 1$. $|\psi(q, t)|^2$ is the probability density at q at time t .

Schrodinger's equation:

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial q^2} + V(q)\psi(q, t) = H\psi$$

where $V(q)$ is the potential function, and $H = -\frac{1}{2} \frac{\partial^2}{\partial q^2} + V$ is the energy (Hamiltonian) function.

Solution to Schrodinger's equation:

$$\psi(\cdot, t) = e^{-itH}\psi(\cdot, 0)$$

where e^{-itH} is unitary because H is symmetric (Hermitian?)

Feynman's path integral solution: "integrate over function space".

$$\psi(q, t) = \int K_t(q, q')\psi(q', 0) dq'$$

where the Schrodinger kernel

$$K_t(q_0, q_1) = \int \exp \left[i \left(\frac{1}{2} \int_0^T \dot{q}(t)^2 dt - \int_0^T V(q(t)) dt \right) \right] Dq$$

where Dq is the "Lebesgue measure" on paths (might not exist rigorously.) The integral is over all paths such that $q(0) = q_0, q(T) = q_1$.

Mark Kac developed the real version of path integral solution, for the heat equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{2} \frac{\partial^2 \psi}{\partial q^2} - V\psi = -H\psi$$

Solutions are

$$\psi(\cdot, t) = e^{-tH}\psi(\cdot, 0)$$

Kac proved rigorously that indeed

$$\psi(q, t) = \int K_t(q, q')\psi(q', 0) dq'$$

where

$$K_t(q_0, q_1) = \int \exp \left(-\frac{1}{2} \int_0^T \dot{q}(t)^2 dt - \int_0^T V(q(t)) dt \right) Dq.$$

Although Dq is still undefined, $\exp(-\frac{1}{2} \int_0^T \dot{q}(t)^2 dt) Dq$ is well-defined: it is the probability law of Brownian motion.

Take any path $q(t), 0 \leq t \leq \tau$, imagine q has analytic continuation to the imaginary axis. Wick rotation trick: $q_1(t) := q(it)$.

$$i \int_0^T \left[\frac{1}{2} \dot{q}(t)^2 - V(q(t)) \right] dt = \int_0^T \left[\frac{1}{2} \dot{q}(it)^2 - V(q(it)) \right] dt$$

2 particle in 1D

Let $\psi(q_1, q_2, t)$ be the quantum state of 2 particles in 1D. Schrodinger equation becomes

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \partial_{q_1}^2 \psi - \frac{1}{2} \partial_{q_2}^2 \psi + V(q_1, q_2) \psi(q_1, q_2, t)$$

For example, $V(q_1, q_2)$ can be $q_1^2 + q_2^2 + (q_1 - q_2)^2$.

Feynman representation of the Schrodinger kernel is similarly

$$\int \exp \left[i \left(\frac{1}{2} \int_0^T \dot{q}_1(t)^2 dt + \frac{1}{2} \int_0^T \dot{q}_2(t)^2 dt - \int_0^T V(q(t)) dt \right) \right] Dq$$

Consider n particles in 1D, each by itself will behave like a simple harmonic oscillator, but there is some weak interaction between any pair, i.e. $V(q_1, q_2, \dots, q_n) = \sum q_i^2 + \epsilon \sum (q_i - q_{i+1})^2$.

QFT deals with infinitely many particles, one at each $x \in \mathbb{R}$. (We are considering 1 space dimension and 1 time dimension.) Then $q_1(t), q_2(t), \dots$ becomes a function $q(x, t)$.

$$q = q_t : \mathbb{R} \rightarrow \mathbb{R}, V(q) = \int q(x)^2 dx + \int (\partial_x q)^2 dx \text{ or } V(q) = \int U(q(x)) dx + \int (\partial_x q)^2 dx$$

It is impossible to write down the Schrodinger PDE. However, Feynman integral can still be written down:

$$\int \exp \left[i \left(\frac{1}{2} \int_{\mathbb{R}} \int_0^T (\partial_t q)^2 dt dx - \int_{\mathbb{R}} \int_0^T U(q(x, t)) dt dx - \frac{1}{2} \int_{\mathbb{R}} \int_0^T (\partial_x q)^2 dt dx \right) \right] Dq$$

integral is over the space of surfaces. (NOTE: where did the last 1/2 come from?)

Wick rotation, $t \rightarrow it$ gives

$$\int \exp \left(-\frac{1}{2} \int_{\mathbb{R}} \int_0^T (\partial_t q)^2 dt dx - \int_{\mathbb{R}} \int_0^T U(q(x, t)) dt dx - \frac{1}{2} \int_{\mathbb{R}} \int_0^T (\partial_x q)^2 dt dx \right) Dq$$

which is exactly the 2D Gaussian free field.

Constructible QFT

Arthur Wightman gave a list of criteria that allow you to do the analytic continuation rigorously, and built an actual QFT. Simplification was given by Osterwalder and Schrader.

Yang-Mill theory is the limit of lattice gauge theory. But QFT is hard to construct. Many of them are Gaussian, i.e. "trivial". It has connection with random surface theory, and Liouville quantum gravity.

Lecture 14: May 17

Lecturer: Sourav Chatterjee

Scriber: Sicong (Scott) Zhang

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Physics Motivation

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A state is $\psi(q, t) \in L^2(\mathbb{R})$ with $\|\psi\|_{L^2} = 1$. $|\psi(q, t)|^2$ is the probability density at q at time t .

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$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial q^2} + V(q)\psi(q, t) = H\psi$$

where $V(q)$ is the potential function, and $H = -\frac{1}{2} \frac{\partial^2}{\partial q^2} + V$ is the energy (Hamiltonian) function.

Solution to Schrodinger's equation:

$$\psi(\cdot, t) = e^{-itH}\psi(\cdot, 0)$$

where e^{-itH} is unitary because H is symmetric (Hermitian?)

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$$\psi(q, t) = \int K_t(q, q')\psi(q', 0) dq'$$

where the Schrodinger kernel

$$K_t(q_0, q_1) = \int \exp \left[i \left(\frac{1}{2} \int_0^T \dot{q}(t)^2 dt - \int_0^T V(q(t)) dt \right) \right] Dq$$

where Dq is the "Lebesgue measure" on paths (might not exist rigorously.) The integral is over all paths such that $q(0) = q_0, q(T) = q_1$.

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where

$$K_t(q_0, q_1) = \int \exp \left(-\frac{1}{2} \int_0^T \dot{q}(t)^2 dt - \int_0^T V(q(t)) dt \right) Dq.$$

Although Dq is still undefined, $\exp(-\frac{1}{2} \int_0^T \dot{q}(t)^2 dt) Dq$ is well-defined: it is the probability law of Brownian motion.

Take any path $q(t), 0 \leq t \leq \tau$, imagine q has analytic continuation to the imaginary axis. Wick rotation trick: $q_1(t) := q(it)$.

$$i \int_0^T [\frac{1}{2} \dot{q}(t)^2 - V(q(t))] dt = \int_0^T [\frac{1}{2} \dot{q}(it)^2 - V(q(it))] dt$$

Finitely many particle in 1D

Let $\psi(q_1, q_2, t)$ be the quantum state of 2 particles in 1D. Schrodinger equation becomes

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \partial_{q_1}^2 \psi - \frac{1}{2} \partial_{q_2}^2 \psi + V(q_1, q_2) \psi(q_1, q_2, t)$$

For example, $V(q_1, q_2)$ can be $q_1^2 + q_2^2 + (q_1 - q_2)^2$.

Feynman representation of the Schrodinger kernel is similarly

$$\int \exp \left[i \left(\frac{1}{2} \int_0^T \dot{q}_1(t)^2 dt + \frac{1}{2} \int_0^T \dot{q}_2(t)^2 dt - \int_0^T V(q(t)) dt \right) \right] Dq$$

Consider n particles in 1D, each by itself will behave like a simple harmonic oscillator, but there is some weak interaction between any pair, i.e. $V(q_1, q_2, \dots, q_n) = \sum q_i^2 + \epsilon \sum (q_i - q_{i+1})^2$.

infinitely many particles

QFT deals with infinitely many particles, one at each $x \in \mathbb{R}$. (We are considering 1 space dimension and 1 time dimension.) Then $q_1(t), q_2(t), \dots$ becomes a function $q(x, t)$.

$$q = q_t : \mathbb{R} \rightarrow \mathbb{R}, V(q) = \int q(x)^2 dx + \int (\partial_x q)^2 dx \text{ or } V(q) = \int U(q(x)) dx + \int (\partial_x q)^2 dx$$

It is impossible to write down the Schrodinger PDE. However, Feynman integral can still be written down:

$$\int \exp \left[i \left(\frac{1}{2} \int_{\mathbb{R}} \int_0^T (\partial_t q)^2 dt dx - \int_{\mathbb{R}} \int_0^T U(q(x, t)) dt dx - \frac{1}{2} \int_{\mathbb{R}} \int_0^T (\partial_x q)^2 dt dx \right) \right] Dq$$

integral is over the space of surfaces. (NOTE: where did the last 1/2 come from?)

Wick rotation, $t \rightarrow it$ gives

$$\int \exp \left(-\frac{1}{2} \int_{\mathbb{R}} \int_0^T (\partial_t q)^2 dt dx - \int_{\mathbb{R}} \int_0^T U(q(x, t)) dt dx - \frac{1}{2} \int_{\mathbb{R}} \int_0^T (\partial_x q)^2 dt dx \right) Dq$$

which is exactly the 2D Gaussian free field.

Constructible QFT

Arthur Wightman gave a list of criteria that allow you to do the analytic continuation rigorously, and built an actual QFT. Simplification was given by Osterwalder and Schrader.

Yang-Mill theory is the limit of lattice gauge theory. But QFT is hard to construct. Many of them are Gaussian, i.e. “trivial”. It has connection with random surface theory, and Liouville quantum gravity.

Lecture 15: May 22

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Disclaimer: These notes may contain factual and/or typographic errors.

15.8 Hermite Polynomials & the Wick-ordered Power

First we'll define the Hermite polynomials $H_k(x)$.

Definition 31 For $k \geq 1$, the k -th Hermite polynomial H_k is given by

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} (e^{-x^2/2})$$

Hermite polynomials can also be defined without the factor of $\frac{1}{2}$ in the the exponential.

Example 5

$$H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3$$

Let γ be the standard Gaussian measure on \mathbb{R} , i.e. with density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ with respect to Lebesgue measure. For $f, g \in L^2(\gamma)$, define

$$(f, g) = \int fg d\gamma$$

Then $(f, H_k) = (f', H_{k-1})$ for $f \in L^2(\gamma)$:

$$\begin{aligned} (f, H_k) &= \int f H_k d\gamma \\ &= \int_{-\infty}^{\infty} f(x) (-1)^k e^{x^2/2} \frac{d^k}{dx^k} (e^{-x^2/2}) \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (-1)^k \frac{d^k}{dx^k} (e^{-x^2/2}) dx \\ &= \int_{-\infty}^{\infty} f'(x) (-1)^{k-1} e^{x^2/2} \frac{d^{k-1}}{dx^{k-1}} (e^{-x^2/2}) \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= (f', H_{k-1}) \end{aligned}$$

Since H_k is a polynomial with leading term x^k , if $j < k$, the above shows that

$$(H_j, H_k) = (H'_j, H_{k-1}) = \cdots = (j!, H_{k-j}) = 0$$

Thus $\{H_k\}$ is a system of orthogonal polynomials for the inner product defined by γ ; moreover, it is a complete basis of $L^2(\gamma)$ (because of density of polynomials in $L^2(\gamma)$).

Now, suppose (X, Y) is a jointly normal random vector with mean 0, variance 1, and variance ρ .

Claim 32 Y can be written as $\rho X + \sqrt{1 - \rho^2} Z$ for X, Z i.i.d $\sim \mathcal{N}(0, 1)$.

This claim will not be proven. Suppose X, Y is jointly Gaussian with mean 0, $\text{Var}(X) = \sigma_X^2$, $\text{Var}(Y) = \sigma_Y^2$, and $\text{Cov}(X, Y) = \sigma_{XY}$.

Definition 33 The Wick-ordered k -th power of X is denoted $:X^k:$, and given by

$$:X^k: := \frac{\sigma_X^k}{\sqrt{k!}} H_k \left(\frac{X}{\sigma_X} \right)$$

Example 6

$$\begin{aligned} :X^4: &:= \frac{\sigma_X^4}{\sqrt{24}} \left(\left(\frac{X}{\sigma_X} \right)^4 - 6 \left(\frac{X}{\sigma_X} \right)^2 + 3 \right) \\ &= \frac{1}{\sqrt{24}} (X^4 - 6\sigma_X^2 X^2 + 3\sigma_X^4) \end{aligned}$$

For any k , $\mathbb{E}[:X^k:] = 0$, and

$$\text{Cov}(:X^k:, :Y^k:) = \frac{\sigma_X^k \sigma_Y^k}{(\sqrt{k!})^2} \mathbb{E} \left[H_k \left(\frac{X}{\sigma_X} \right) H_k \left(\frac{Y}{\sigma_Y} \right) \right] = \sigma_X^k \sigma_Y^k \rho_{XY}^k$$

where

$$\rho_{XY} = \text{Cov} \left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y} \right) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Thus $\text{Cov}(:X^k:, :Y^k:) = \sigma_{XY}^k$.

15.9 ϕ^4 theory on the 2D torus

We will now see a sketch of the construction of ϕ^4 theory on the 2D torus. Consider an ϵ -discretization of the torus $[0, 1]^2 \cap \epsilon\mathbb{Z}^2$.

Let ϕ be a Gaussian free field on this grid with mass $m\epsilon$, i.e. with density proportional to

$$\exp \left(-\frac{1}{2} \sum_{\langle uv \rangle} (\phi_u - \phi_v)^2 - \frac{m^2 \epsilon^2}{2} \sum_u \phi_u^2 \right)$$

which converges to the GFF. Let $\sigma\epsilon^2 = \text{Var}(\phi_u) \sim C \log(\frac{1}{\epsilon})$. Let $V(\phi) = \lambda\epsilon^2 \sum_u : \phi_u^4 :$. Then

$$V(\phi) = \lambda\epsilon^2 \sum_u (\phi_u^4 - 6\sigma_\epsilon^2 \phi_u^2 + 4\sigma_\epsilon^4)$$

Consider the probably measure μ_ϵ with density proportional to

$$\exp \left(-\frac{1}{2} \sum_{\langle uv \rangle} (\phi_u - \phi_v)^2 - \frac{m^2 \epsilon^2}{2} \sum_u \phi_u^2 - V(\phi) \right)$$

which is in turn proportional to

$$\exp\left(-\frac{1}{2}\sum_{\langle uv\rangle}(\phi_u - \phi_v)^2 + (6\lambda\sigma_\epsilon^2 - \frac{m^2}{2})\epsilon^2\sum_u\phi_u^2 - \lambda\epsilon^2\sum_u\phi_u^4\right)$$

We will show that there is a convergent subsequence of these measures. To do this, we'll prove "ultraviolet stability", which says that for all $\alpha > 0$, there exist $c_1, c_2 > 0$ dependent only on m, λ, α such that

$$c_1 \leq \mathbb{E}[\epsilon^{-V(\phi)}] \leq c_2 \quad (15.25)$$

for $\phi \sim$ GFF at scale ϵ .

For any nonnegative function F on the space of configurations,

$$\int F d\mu_\epsilon = \frac{\mathbb{E}[F(\phi)\epsilon^{-V(\phi)}]}{\mathbb{E}[\epsilon^{-V(\phi)}]} \leq \frac{\sqrt{\mathbb{E}[F(\phi)^2]\mathbb{E}[\epsilon^{-2V(\phi)}]}}{\mathbb{E}[\epsilon^{-V(\phi)}]} \leq \text{const.} \cdot \sqrt{\mathbb{E}[F(\phi)^2]}$$

So if F is a function such that $\mathbb{E}[F(\phi)^2]$ is uniformly bounded, irrespective of ϵ , so is $\int F d\mu$. This can be used to prove the tightness of linear functionals under μ .

The same inequality can be used to prove uniform equicontinuity of characteristic functions under μ ; then, one can get a subsequential limit by Arzela-Ascoli.

By the definition of $V(\phi)$ as $\lambda\epsilon^2\sum_u\phi_u^4$, $\mathbb{E}[V(\phi)] = 0$, and

$$\text{Var}(V(\phi)) = \lambda^2\epsilon^4\sum_{u,v}\text{Cov}(\phi_u^4, \phi_v^4) := \lambda^2\epsilon^4\sum_{u,v}C_\epsilon(u,v)$$

where $C_\epsilon(u,v) = \text{Cov}(\phi_u^4, \phi_v^4)$. This converges to $\lambda^2\int\int C(x,y)^4 dx dy$ where $C(x,y)$ is the covariance kernel of the GFF with mass on the torus.

Remark 34 If $V(\phi)$ were defined as $\lambda\epsilon^2\sum_u\phi_u^4$, rather than using the Wick-ordered power, it would blow up as $\epsilon \rightarrow 0$.

Note that by Jensen's inequality,

$$\mathbb{E}[\epsilon^{-V(\phi)}] \geq \epsilon^{-\mathbb{E}[V(\phi)]} = 1$$

Now suppose X is a multivariate normal random variable with mean 0 and covariance matrix Σ , and suppose Σ can be decomposed as a sum of positive definite matrices Σ_1 and Σ_2 . Then, if $Y \sim \mathcal{N}(0, \Sigma_1)$, $Z \sim \mathcal{N}(0, \Sigma_2)$, with Y and Z independent,

$$Y + Z \stackrel{\mathcal{D}}{=} X$$

One common way to do this kind of decomposition is to write

$$\Sigma = \sum_{i=1}^n \lambda_i u_i u_i^T \quad (\text{spectral decomposition})$$

and take

$$\Sigma_1 = \sum_{i=1}^k \lambda_i u_i u_i^T, \quad \Sigma_2 = \sum_{i=k+1}^n \lambda_i u_i u_i^T$$

The same can be done for the discrete GFF: take $\phi = \phi_k + \phi'_k$, ϕ_k and ϕ'_k independent, where the level k depends on where you truncate. For this decomposition,

$$\begin{aligned} \mathbb{E} \left[\left(\epsilon^2 \sum_u : \phi_u^4 : - \epsilon^2 \sum_u : \phi_{k,u}^4 : \right)^2 \right] &= \epsilon^4 \sum_{u,v} \mathbb{E} [(: \phi_u^4 : - : \phi_{k,u}^4 :) (: \phi_v^4 : - : \phi_{k,v}^4 :)] \\ &= \epsilon^4 \sum_{u,v} [\text{Cov}(\phi_u, \phi_v)^4 + \text{Cov}(\phi_{k,u}, \phi_{k,v})^4] \\ &\quad - \epsilon^4 \sum_{u,v} [\text{Cov}(\phi_u, \phi_{k,v})^4 + \text{Cov}(\phi_v, \phi_{k,u})^4] \end{aligned}$$

Since $\text{Cov}(\phi'_{k,u}, \phi_{k,v}) = 0$ by independence, this simplifies to

$$\mathbb{E} \left[\left(\epsilon^2 \sum_u : \phi_u^4 : - \epsilon^2 \sum_u : \phi_{k,u}^4 : \right)^2 \right] = \epsilon^4 \sum_{u,v} [\text{Cov}(\phi_u, \phi_v)^4 - \text{Cov}(\phi_{k,u}, \phi_{k,v})^4]$$

For each k , we choose the decomposition such that as $\epsilon \rightarrow 0$,

$$\text{Cov}(\phi_{k,u}, \phi_{k,v}) \xrightarrow{u \rightarrow x, v \rightarrow y} C_k(x, y)$$

where $|C - C_k| = O(k^{-\delta})$. Then, for all j , there will be an integrable bound such as:

$$\mathbb{E} \left[\left(\epsilon^2 \sum_u : \phi_u^4 : - \epsilon^2 \sum_u : \phi_{k,u}^4 : \right)^{2j} \right] \leq (C_1 j)^{C_2 j} k^{-\delta j}$$

Lecture 16: May 24

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Disclaimer: These notes may contain factual and/or typographic errors.

16.10 Bounding $\mathbb{E}[\epsilon^{-V(\phi)}]$

Let ϕ a Gaussian free field, ϕ_k a smoothed version of ϕ . We want to show that

$$\mathbb{E}[\epsilon^{-V(\phi)}] \leq C$$

where C is independent of the scaling ϵ , and $V(\phi) = \lambda \epsilon^2 \sum_u \phi_u^4$. We can choose k so that

$$-V(\phi_k) \leq C(\log k)^\alpha$$

and

$$\mathbb{E}[|V(\phi) - V(\phi_j)|^j] \leq (C_1 j)^{C_2 j} k^{-\delta j} \quad \forall j$$

where C_1, C_2, δ are constants, by what we did last time.

Using the moment inequality and Chebyshev's inequality, we get

$$\mathbb{P}(|V(\phi) - V(\phi_k)| \geq 1) \leq \mathbb{E}[|V(\phi) - V(\phi_k)|^j] \leq (C_1 j)^{C_2 j} k^{-\delta j}$$

Choosing $j = k^\epsilon$ for some small enough ϵ , we see that this probability is $\leq C_1 \epsilon^{-C_2 k^\epsilon}$. Thus

$$\begin{aligned} \mathbb{P}(\epsilon^{-V(\phi)} \geq a) &= \mathbb{P}(\epsilon^{-(V(\phi) - V(\phi_k)) - V(\phi_k)} \geq a) \\ &\leq \mathbb{P}(\epsilon^{-(V(\phi) - V(\phi_k))} \epsilon^{C(\log k)^\alpha} \geq a) \end{aligned}$$

Choose k such that $\epsilon^{(C \log k)^\alpha} \sim a$, which means that $k \sim \epsilon^{C'(\log a)^\gamma}$. Then

$$\mathbb{P}(\epsilon^{-V(\phi)} \geq a) \leq \mathbb{P}(|V(\phi) - V(\phi_k)| \geq \text{const.}) \leq C_1 \epsilon^{-C_2 k^\epsilon} = C_1 \epsilon^{-C_2 \epsilon^{C_3 (\log a)^\gamma}}$$

16.11 Decay of Correlations

Recall the $O(2)$ model on the 2D torus $\mathbb{T}_n^2 = \{0, 1, \dots, n-1\}^2$. The $O(2)$ model has spins $\sigma_x \in S^1$ on each $x \in V(\mathbb{T}_n^2)$, $\sigma_x = (\cos \theta_x, \sin \theta_x)$. The probability density is proportional to $\exp(\beta \sum_{\langle xy \rangle} \cos(\theta_x - \theta_y))$.

Claim 35 $\rho_{xy} = \langle \sigma_x \cdot \sigma_y \rangle = \langle \cos(\theta_x - \theta_y) \rangle$ decays like $|x - y|^{-\alpha}$ for some α .

Intuition. Suppose we approximate $\cos(\theta_x - \theta_y)$ by its first order Taylor expansion:

$$\cos(\theta_x - \theta_y) \approx 1 - \frac{1}{2}(\theta_x - \theta_y)^2$$

Then we get back the Gaussian free field. If θ is a configuration from the Gaussian free field on \mathbb{T}_n^2 , then $\cos(\theta_x - \theta_y)$ is a normal random variable, so

$$\langle \cos(\theta_x - \theta_y) \rangle = \epsilon^{-\frac{1}{2}\mathbb{E}[(\theta_x - \theta_y)^2]} \sim |x - y|^{-\alpha}$$

since $\mathbb{E}[(\theta_x - \theta_y)^2] \sim \log |x - y|$.

For small β , there is exponential decay of correlations, so this is not the correct answer. For large β , $\cos(\theta_x - \theta_y)$ tends to 1, so that θ_x, θ_y tend to be close for x, y neighbors.

Patrascioiu-Seiler proposed $O(2)$ model restricted to configurations where $\cos(\theta_x - \theta_y) \geq \frac{1}{\sqrt{2}}$ for all $\langle xy \rangle$. The idea is that θ_x, θ_y “close enough” is sufficient for a power law decay. Aizenman proved power law decay lower bound for this model.

Theorem 36 *For this model, for any ℓ ,*

$$\max_{\|x-y\|_1 \geq \ell} \rho_{xy} \geq \frac{1}{2\ell^2}$$

Proof: Write $\sigma_x = (\sigma_x^1, \sigma_x^2)$. Then, since e.g. $\sigma_x^1 = |\sigma_x^1| \epsilon_x$ where $\epsilon_x = \text{sgn}(\sigma_x^1)$,

$$\begin{aligned} \rho_{xy} &= \mathbb{E}[\sigma_x^1 \sigma_y^1 + \sigma_x^2 \sigma_y^2] \\ &= 2\mathbb{E}[\sigma_x^1 \sigma_y^1] \\ &= 2\mathbb{E}[|\sigma_x^1| |\sigma_y^1| \epsilon_x \epsilon_y] \\ &= 2\mathbb{E}[|\sigma_x^1| |\sigma_y^1| \mathbb{E}[\epsilon_x \epsilon_y | \sigma^2]] \end{aligned}$$

for $\sigma^2 = (\sigma_x^2)_{x \in \mathbb{T}_n^2}$. Let $U: [-1, 1] \rightarrow (-\infty, \infty]$,

$$U(x) = \begin{cases} \infty & \text{if } x < \frac{1}{\sqrt{2}} \\ -\beta x & \text{if } x \geq \frac{1}{\sqrt{2}} \end{cases}$$

Then the density of σ is proportional to $\exp(-\sum_{\langle uv \rangle} U(\sigma_u \cdot \sigma_v))$. This can be written as

$$\exp\left(-\sum_{\langle uv \rangle} U(|\sigma_u^1| |\sigma_v^1| \epsilon_u \epsilon_v + \sigma_u^2 \sigma_v^2)\right)$$

So if

$$f_{uv}(\epsilon) = \exp\left(-\sum_{\langle uv \rangle} U(|\sigma_u^1| |\sigma_v^1| \epsilon_u \epsilon_v + \sigma_u^2 \sigma_v^2) + \sum_{\langle uv \rangle} U(-|\sigma_u^1| |\sigma_v^1| + \sigma_u^2 \sigma_v^2)\right) - 1,$$

then the conditional density of ϵ given σ^2 is proportional to $\prod_{\langle uv \rangle} (f_{uv}(\epsilon) + 1)$.

Note that U is a nonincreasing function, which implies that $f_{uv}(\epsilon) \geq 0$. Moreover, if $\epsilon_u \neq \epsilon_v$, then $f_{uv}(\epsilon) = 0$; if $\epsilon_u = \epsilon_v$, then $f_{uv}(\epsilon)$ depends only on σ^2 .

The conditional density of ϵ given σ^2 can be written as

$$\frac{1}{Z} \sum_{E \subset \mathbb{E}[\mathbb{T}_n^2]} \prod_{\langle uv \rangle \in E} f_{uv}(\epsilon) \times \frac{1}{Z(E)} \times Z(E)$$

where

$$Z(E) = \sum_{\epsilon \in \{\pm 1\}^{V(\mathbb{T}_n^2)}} \prod_{\langle uv \rangle \in E} f_{uv}(\epsilon)$$

This model can be reinterpreted as follows: pick a set of edges E with probability $Z(E)/Z$. Given E , pick ϵ with density proportional to $\prod_{\langle uv \rangle \in E} f_{uv}(\epsilon)$.

Note that given E , we get a division of $V(\mathbb{T}_n^2)$ into connected components. Any configuration ϵ where ϵ_u is not the same everywhere within a connected component has probability 0. Moreover, any allowed configuration has the same probability. Thus, given E , ϵ is uniformly distributed on the set of all configurations where ϵ is constant in each connected component. Therefore

$$\mathbb{E}[\epsilon_x \epsilon_y | \sigma^2, E] = \mathbf{1}_{\{x \leftrightarrow y \in E\}}$$

This shows in particular that ρ_{xy} is nonnegative, even in the original $O(2)$ model—we have not used yet that $\cos(\theta_x - \theta_y) \geq \frac{1}{\sqrt{2}}$.

Let $V_0 = \{v \in V(\mathbb{T}_n^2) : |\sigma_v^1| \geq \frac{1}{\sqrt{2}}\}$. Define a new adjacency structure on \mathbb{T}_n^2 by adding edges along the diagonals of the lattice. Then if $u, v \in V_0$ and u, v are neighbors in the new graph, then $\epsilon_u = \epsilon_v$, since the angle between neighbors in the new graph is at most $\pi/2$.

Let $E_{xy} = \{x \leftrightarrow y \in V_0 \text{ with new adjacency structure}\}$.

E_{xy} depends only on σ^2 . E_{xy} true implies $\epsilon_x = \epsilon_y$, which in turn implies that x must be connected to y in the graph determined by E . Thus, $\mathbb{E}[\epsilon_x \epsilon_y | \sigma^2, E] \geq \mathbf{1}_{E_{xy}}$. So

$$\rho_{xy} = \mathbb{E}[\sigma_x \cdot \sigma_y] = 2\mathbb{E}[|\sigma_x^1| |\sigma_y^1| \mathbb{E}[\epsilon_x \epsilon_y | \sigma^2]] \geq 2\mathbb{E}[|\sigma_x^1| |\sigma_y^1| \mathbf{1}_{E_{xy}}] \geq \mathbb{P}(E_{xy})$$

Let $R = \{1, \dots, \ell\}^2 \subset V(\mathbb{T}_n^2)$, and

$$A = \{\exists \text{ top-bottom crossing of } R \text{ w. vertices in } V_0 \text{ in new adj. structure}\},$$

$$B = \{\exists \text{ left-right crossing of } R \text{ w. vertices not in } V_0 \text{ in old adj. structure}\}.$$

Fact: either A or B must happen. Thus $1 \leq \mathbb{P}(A) + \mathbb{P}(B)$. But by $\sigma^1 \rightarrow \sigma^2$ symmetry,

$$\mathbb{P}(B) = \mathbb{P}(\tilde{B})$$

where

$$\tilde{B} = \{\exists \text{ left-right crossing of } R \text{ in old adj. through } V_0\}$$

$\mathbb{P}(\tilde{B}) \leq \mathbb{P}(A)$, since the new adj. has more edges. Thus

$$1 \leq 2\mathbb{P}(A) \leq 2 \sum_{x \in \text{top of } R, y \in \text{bottom of } R} \mathbb{P}(E_{xy})$$

■

Lecture 17: June 5

Lecturer: Sourav Chatterjee

Scriber: Song Mei

Disclaimer: These notes may contain factual and/or typographic errors.

In this lecture, we are talking about Liouville Quantum Gravity.

17.12 Gaussian Free Field

Let \mathcal{D} be a bounded domain in \mathbb{R}^2 . We would like to consider a Gaussian free field (GFF) on \mathcal{D} with zero boundary condition. One way to define this GFF is to approximate \mathcal{D} by a grid and then take the grid spacing to zero. The limiting object is a random distribution h on \mathcal{D} .

Definition 37 (Gaussian Free Field) We say h is a Gaussian Free Field on \mathcal{D} with zero boundary condition, if for any smooth compactly supported f on \mathcal{D} , we have

$$\int_{\mathcal{D}} fh \sim \mathcal{N}\left(0, \int_{\mathcal{D}} \int_{\mathcal{D}} f(x)f(y)G_{\mathcal{D}}(x,y)\delta x\delta y\right), \quad (17.26)$$

where $G_{\mathcal{D}}(x,y)$ is Green's function on \mathcal{D} with Dirichlet boundary condition. This means, we have

$$G_{\mathcal{D}}(x,y) = \mathbb{E}\left[\int_0^\infty p_{\mathcal{D}}(t,x,y)\delta t\right], \quad (17.27)$$

where $p_{\mathcal{D}}(t,x,y)$ is the transition kernel of a Brownian motion killed when hitting $\partial\mathcal{D}$.

An alternative representation of $G_{\mathcal{D}}$ gives: Consider the Laplacian Δ acting on smooth functions on \mathcal{D} that vanishes at the boundary. Under mild condition on \mathcal{D} , there exists an $L^2(\overline{\mathcal{D}})$ orthonormal sequences $\{u_n\}_{n \in \mathbb{N}_+}$ on \mathcal{D} , where each u_n is smooth and vanishes at the boundary such that u_n is an eigenfunction of Δ , and $\{u_n\}_{n \in \mathbb{N}_+}$ is a basis of $L^2(\overline{\mathcal{D}})$. Then the Green's function can be formally represented as

$$G_{\mathcal{D}}(x,y) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} u_n(x)u_n(y), \quad x \neq y, \quad (17.28)$$

where

$$\Delta u_n = -\lambda_n u_n. \quad (17.29)$$

Also, the GFF can be formally represented as

$$h(x) = \sum_{n=1}^{\infty} \frac{X_n}{\sqrt{\lambda_n}} u_n(x), \quad (17.30)$$

where $X_i \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

As $d \geq 2$, this sum diverges. For example, $\mathcal{D} = (0,1)^d$, then the eigenfunctions are $\prod_{j=1}^d \sin(\pi k_j x_j)$ indexed by (k_1, \dots, k_d) . The eigenvalues are $\lambda(k_1, \dots, k_d) = \pi^2(k_1^2 + \dots + k_d^2)$. Alternatively using n as indices, we have $\lambda_n \approx n^{2/d}$.

But for Equation (17.30), the convergence is still valid on various spaces of distributions.

17.13 Domain Markov Properties

Proposition 38 *Let $\mathcal{U} \subset \mathcal{D}$. Let h be a GFF on \mathcal{D} with zero boundary condition. Then h can be written as $h = h_0 + \varphi$, with*

1. h_0 is independent of φ .
2. h_0 is a GFF on \mathcal{U} with zero boundary condition and $h_0 = 0$ on $\mathcal{D} \setminus \mathcal{U}$.
3. φ is harmonic on \mathcal{U} .

For the discrete GFF, Markovian property is obvious from the definition. Moreover, given the value on the boundary of a sub-graph \mathcal{U} , the field inside has the same covariance structure as a GFF with zero boundary condition, but the mean is a discrete harmonic function with the prescribed boundary values.

Thus, inside \mathcal{U} , the conditioned distribution of the GFF given the values on $\partial\mathcal{U}$ is the law of a zero bounded GFF on \mathcal{U} plus a harmonic function with given boundary values.

One can get the continuum decomposition by passing to the continuum limit from the discrete. But one can also get it by a direct continuum argument.

17.14 Circle Averages

Let h be a GFF on a domain $\mathcal{D} \subset \mathbb{R}^2$ with zero boundary condition. For any $z \in \mathcal{D}$, and $\varepsilon < \text{dist}(z, \partial\mathcal{D})$, let

$$h_\varepsilon(z) = 1/(2\pi\varepsilon) \cdot \int_{\partial\mathbb{B}(z,\varepsilon)} h(x) \delta x. \quad (17.31)$$

Take any $z \in \mathcal{D}$ and $\varepsilon_0 < \text{dist}(z, \partial\mathcal{D})$. For $t \geq t_0 = \log(1/\varepsilon_0)$, let $B_t = h_{e^{-t}}(z)$, then $(B_t)_{t \geq t_0}$ is a standard Brownian motion starting at B_{t_0} .

The reason is that, we take two radius $\varepsilon_1 > \varepsilon_2$. On $\mathbb{B}(z, \varepsilon_1)$, write $h = h_0 + \phi$, with h_0 a GFF with zero boundary in $\mathbb{B}(z, \varepsilon_1)$, and ϕ a harmonic function in $\mathbb{B}(z, \varepsilon_1)$ and agrees with h outside. h_0 and ϕ are independent.

Then

$$h_{\varepsilon_2}(z) = h_{0,\varepsilon_2}(z) + \varphi_{\varepsilon_2}(z), \quad (17.32)$$

$$h_{\varepsilon_1}(z) = h_{0,\varepsilon_1}(z) + \varphi_{\varepsilon_1}(z). \quad (17.33)$$

But $h_{0,\varepsilon_1}(z) = 0$, since h_0 is GFF on $\mathbb{B}(z, \varepsilon_1)$ with zero boundary. Therefore, $h_{\varepsilon_1}(z) = \varphi_{\varepsilon_1}(z)$. Since φ is harmonic in $\mathbb{B}(z, \varepsilon_1)$, $\varphi_{\varepsilon_1}(z) = \phi_{\varepsilon_2}(z)$. Therefore $h_{\varepsilon_2}(z) = h_{0,\varepsilon_2}(z) + h_{\varepsilon_1}(z)$. Therefore $h_{\varepsilon_2}(z) - h_{\varepsilon_1}(z) = h_{0,\varepsilon_2}(z)$ which is independent of $\varphi_{\varepsilon_1}(z) = h_{\varepsilon_1}(z)$.

Therefore, we proved that $h_{e^{-t}}(z)$ as a random function of t has independent increment. The parameterization using t makes the variance exactly to be t .

Definition 39 (Thick points) *A point z is called α -thick for a GFF h if*

$$\liminf_{\varepsilon \rightarrow 0} h_\varepsilon(z) / \log(1/\varepsilon) = \alpha. \quad (17.34)$$

It is known that a 2D GFF has α -thick points for $\forall \alpha \in [0, 2)$. Moreover, $\dim(\text{Set of } \alpha \text{ thick points}) = 2 - \alpha^2/2$.

17.15 Liouville Quantum Gravity

Definition 40 *LQG on a bounded domain $\mathcal{D} \subset \mathbb{R}^2$ is a Riemannian metric on \mathcal{D} , which is proportional to*

$$\begin{bmatrix} \exp(\gamma h(z)) & 0 \\ 0 & \exp(\gamma h(z)) \end{bmatrix} \quad (17.35)$$

at z .

The volume form for this metric is proportional to $\exp(\gamma h(z))\delta z$. This made rigorous by Duplantier and Sheffield.

Let $\mu_\varepsilon(\delta z) = \exp(\gamma h_\varepsilon(z) - \gamma^2/2 \cdot \log(1/\varepsilon))\delta z$. We have the following theorem

Theorem 41 *μ_ε has a weak limit μ as $\varepsilon \rightarrow 0$ though 2^{-k} . As $\text{Leb}(A) > 0$, we have $\mu(A) > 0$. We also have $\mu(\mathcal{D}) < \infty$.*

We will prove this theorem in the next lecture.

Lecture 18: June 7

Lecturer: Sourav Chatterjee

Scriber: Song Mei

Disclaimer: *These notes may contain factual and/or typographic errors.*

In this lecture, we are talking about Liouville Quantum Gravity.

18.16 Liouville Quantum Gravity

Definition 42 *LQG on a bounded domain $\mathcal{D} \subset \mathbb{R}^2$ is a Riemannian metric on \mathcal{D} , which is proportional to*

$$\begin{bmatrix} \exp(\gamma h(z)) & 0 \\ 0 & \exp(\gamma h(z)) \end{bmatrix} \quad (18.36)$$

at z .

The volume form for this metric is proportional to $\exp(\gamma h(z))\delta z$. This made rigorous by Duplantier and Sheffield.

Let $\mu_\varepsilon(\delta z) = \exp(\gamma h_\varepsilon(z) - \gamma^2/2 \cdot \log(1/\varepsilon))\delta z$. We have the following theorem

Theorem 43 *μ_ε has a weak limit μ as $\varepsilon \rightarrow 0$ through 2^{-k} . As $\text{Leb}(A) > 0$, we have $\mu(A) > 0$. We also have $\mu(\mathcal{D}) < \infty$.*

18.17 Proof

Let h be a 2D Gaussian Free Field on a bounded domain $\mathcal{D} \subset \mathbb{R}^2$. Let $h_\varepsilon(z)$ be the average of h in a circle of radius ε around z . Let $B_t = h_{e^{-t}}(z)$. Then B_t is a standard Brownian motion. Let $\mu_\varepsilon(\delta z) = \exp\{\gamma h_\varepsilon(z) - \gamma^2/2 \cdot \log(1/\varepsilon)\}\delta z$. Claim: μ_ε converges to a random limit measure μ , if $\gamma \in [0, 2)$. This μ is the volume form of $LQG(\gamma)$.

We will show that for any Borel set $A \subset \mathcal{D}$, $\mu_\varepsilon(A)$ converges to a finite limit almost surely as $\varepsilon \rightarrow 0$ through 2^{-k} .

The proof is easier when $\gamma < \sqrt{2}$. We have

$$\mu_\varepsilon(A) = \int_A \exp\{\gamma h_\varepsilon(z) - \gamma^2/2 \cdot \log(1/\varepsilon)\}\delta z. \quad (18.37)$$

A direct calculation shows that

$$\mathbb{E}[(\mu_\varepsilon(A) - \mu_{\varepsilon/2}(A))^2] \leq C\varepsilon^{2-\gamma^2}. \quad (18.38)$$

So if $\gamma < \sqrt{2}$, then $\{\mu_{2^{-k}}(A)\}_{k \geq 1}$ is almost surely Cauchy.

The proof for all $\gamma \in [0, 2)$ is more complicated. For some $\varepsilon_0 > 0$, some $\alpha > \gamma$, let

$$G_\varepsilon(z) = \{h_\gamma(z) \leq \alpha \log(1/\gamma), \forall \gamma \in [\varepsilon, \varepsilon_0]\}. \quad (18.39)$$

Let $I_\varepsilon = \mu_\varepsilon(A)$, and $J_\varepsilon = \int_A \mathbf{1}\{G_\varepsilon(z)\} \delta \mu_\varepsilon(z)$. We claim that

1. $\sup_\varepsilon \mathbb{E}[J_\varepsilon^2] < \infty$,
2. $\mathbb{E}[e^{\bar{h}_\varepsilon(z)}(1 - \mathbf{1}\{G_\varepsilon(z)\})] \leq C(\varepsilon_0), \forall z$, where $C(\varepsilon_0) \rightarrow 0$ as $\varepsilon_0 \rightarrow 0$, and $\bar{h}_\varepsilon(z) = \gamma h_\varepsilon(z) - \gamma^2/2 \cdot \log(1/\varepsilon)$.

Observation: (1) and (2) imply that $\{I_\varepsilon\}_{\varepsilon > 0}$ is uniformly integrable. Actually, take any $\delta > 0$. Find ε_0 so small that $C(\varepsilon_0) \leq \delta/100$. Then (2) implies that $\mathbb{E}\lambda I_\varepsilon - J_\varepsilon \lambda \leq \delta/100$. (1) implies that $\{J_\varepsilon\}_{\varepsilon > 0}$ is uniformly integrable. Find k so that $\mathbb{E}[\lambda J_\varepsilon \lambda; \lambda J_\varepsilon \lambda > k] \leq \delta/100$.

Using these, we get a uniform bound on $\mathbb{E}[\lambda J_\varepsilon \lambda; \lambda J_\varepsilon \lambda > k]$. Uniformly integrability and a martingale argument gives the convergence of I_ε .

Therefore

$$\mathbb{E}[e^{\bar{h}_\varepsilon(z)}(1 - \mathbf{1}\{G_\varepsilon(z)\})] = 1 - \tilde{P}(G_\varepsilon(z)), \quad (18.40)$$

$$\frac{\delta \tilde{P}}{\delta P}(z) = e^{\bar{h}_\varepsilon(z)}. \quad (18.41)$$

We have

$$\mathbb{E}[e^{\gamma B_T - \gamma^2/2 \cdot T} f((B_t)_{0 \leq t \leq T})] = \mathbb{E}[f((\tilde{B}_t)_{0 \leq t \leq T})], \quad (18.42)$$

where $\tilde{B}_t = B_t + \gamma t$.

Proof of (1) sketch:

$$\mathbb{E}[J_\varepsilon^2] = \int_{A \times A} \mathbb{E}[e^{\bar{h}_\varepsilon(z) + \bar{h}_\varepsilon(y)} \mathbf{1}\{G_\varepsilon(z) \cap G_\varepsilon(y)\}] \delta x \delta y \quad (18.43)$$

$$= \int_{A \times A} \mathbb{E}[e^{\bar{h}_\varepsilon(z) + \bar{h}_\varepsilon(y)}] \tilde{\mathbb{P}}(G_\varepsilon(x) \cap G_\varepsilon(y)) \delta x \delta y \quad (18.44)$$

$$\leq (\varepsilon')^{-\gamma^2} \tilde{\mathbb{P}}(h_\varepsilon(x) \leq \alpha \log(1/\varepsilon)). \quad (18.45)$$

where $\varepsilon' = \max\{\lambda x - y\lambda, \varepsilon\}$. Under $\tilde{\mathbb{P}}$, $h_\varepsilon(z) \sim \mathcal{N}(2r \log(1/\varepsilon), \log(1/\varepsilon))$. Therefore

$$\mathbb{E}[J_\varepsilon^2] \leq C \int_{A \times A} \lambda x - y\lambda^{-\gamma^2 + (2\gamma - \alpha)^2/2} \delta x \delta y. \quad (18.46)$$

If $\gamma < 2$, then we can find $\alpha > r$ such that $(2\gamma - \alpha)^2/2 - \gamma^2 > -2$.

18.18 KPZ relation

Definition 44 Let $A \subset \mathcal{D}$. The Euclidean scaling exponent of A gives

$$x = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}(B(z, \varepsilon) \cap A \neq \emptyset)}{\log(\varepsilon^2)}. \quad (18.47)$$

where $z \sim \text{Unif}(\mathcal{D})$.

Under some condition on A , we have

$$\dim(A) = 2(1 - x). \tag{18.48}$$

Definition 45 *Let $A \subset \mathcal{D}$. The Quantum scaling exponent of A gives*

$$\Delta = \lim_{\delta \rightarrow 0} \frac{\mathbb{P}(\mathbb{B}_\delta(z) \cap A \neq \emptyset)}{\log(\delta)}. \tag{18.49}$$

where $z \sim \text{Unif}(\mathcal{D})$ and $\mathbb{B}_\delta(z)$ is the ball centered at z that has LQG(z) mass δ .

The KPZ relation gives: for any A , we have $x = \gamma^2/4 \cdot \Delta^2 + (1 - \gamma^2/4)\Delta$.