

# New results for surface growth

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- ▶ In probability theory, we often study the motion of points through space — for example, Brownian motion, diffusion processes, SLE, etc.
- ▶ The study of **surfaces** evolving in time has become popular in recent years.
- ▶ **Static** random surfaces have received much attention over the last twenty years; I will not talk about that in this talk. The main focus of this talk will be **dynamically evolving** random surfaces.

# Growing random surfaces

- ▶ A  $d$ -dimensional growing random surface is a random function  $f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d \rightarrow \mathbb{R}$ , where  $f(t, x)$  denotes the height of the surface at location  $x$  at time  $t$ .
- ▶ The time parameter can also be continuous; that is, the domain of  $f$  can be  $\mathbb{R}_{\geq 0} \times \mathbb{Z}^d$ .
- ▶ The simplest example is the [random deposition model](#), where the height  $f(t, x)$  at each  $x$  grows as a random walk with i.i.d. increments, independently of the heights at other locations.
- ▶ In this model,  $\text{Var}(f(t, x))$  grows like a multiple of  $t$ .
- ▶ Complexities arise when the increase in  $f(t, x)$  has some dependence on the heights at neighboring points.

# Edwards–Wilkinson growth

- ▶ In the (discrete time) **Edwards–Wilkinson model**,

$$f(t+1, x) = \frac{1}{2d} \sum_{y \sim x} f(t, y) + z_{t+1, x},$$

where  $y \sim x$  means  $y$  is a neighbor of  $x$ , and  $\{z_{t,x} : t \in \mathbb{Z}_{\geq 0}, x \in \mathbb{Z}^d\}$  are i.i.d. random variables.

- ▶ This is also a very simple model, only slightly more complex than random deposition.
- ▶ Here  $f(t, x)$  can be expressed as a linear combination of the  $z$  variables, which are i.i.d.
- ▶ Using that representation, it is not hard to show that  $\text{Var}(f(t, x))$  grows like a multiple of  $t^{1/2}$  if  $d = 1$ ,  $\log t$  if  $d = 2$ , and converges to a constant as  $t \rightarrow \infty$  if  $d \geq 3$ .

# Last-passage percolation

- ▶ The random surface generated by the **last-passage percolation model** grows as

$$f(t+1, x) = \max_{y \sim x} f(t, y) + z_{t+1, x},$$

where, again, the  $z$ 's are i.i.d. random variables.

- ▶ This is a much more complex model than the previous two.
- ▶ We now know a lot about this model when  $d = 1$  and the  $z$ 's are i.i.d. exponential random variables.
- ▶ In particular, from the work of Johansson '00 and others, we know that for  $d = 1$  and i.i.d. exponential noise variables,  $\text{Var}(f(t, x))$  behaves like a multiple of  $t^{2/3}$ .
- ▶ Moreover, for some suitable constant  $C$ ,  $(f(t, x) - Ct)/t^{1/3}$  converges in distribution to the **Tracy–Widom law** as  $t \rightarrow \infty$ .
- ▶ The results in this special case rely on our ability to do **exact calculations**, which is sometimes called **integrability**.

## Other models

- ▶ In the preceding slides, we saw three models which are now well-understood.
- ▶ There are several other **one-dimensional models** that are integrable.
- ▶ However, these are exceptions — there are many, many models of surface growth from statistical physics, both in dimension one and beyond, which are far beyond the reach of available mathematical techniques.
- ▶ Some examples:
  - ▶ Ballistic deposition.
  - ▶ Eden model.
  - ▶ Restricted solid-on-solid model.
  - ▶ Last-passage percolation in  $d \geq 2$ .
  - ▶ Any non-trivial model in  $d \geq 2$ .
- ▶ A broader concern is that we do not have anything like a ‘general theory’ of dynamically evolving random surfaces, as we do for stochastic processes (i.e., randomly moving points).

# Scaling limits

- ▶ If  $f$  is a growing random surface as before, a **scaling limit** of  $f$  is obtained by considering the rescaled function

$$f_\varepsilon(t, x) := \varepsilon^\alpha f(\varepsilon^{-\beta} t, \varepsilon^{-\gamma} x)$$

and sending  $\varepsilon \rightarrow 0$ , for some suitable exponents  $\alpha$ ,  $\beta$  and  $\gamma$ .

- ▶ The exponents need to be chosen so that a **nontrivial limit** is obtained.
- ▶ In  $d = 1$ , it is believed (and sometimes proved) that for a large class of nontrivial models, the correct exponents are  $\alpha = 1/2$ ,  $\beta = 3/2$  and  $\gamma = 1$ , and the scaling limit is the **Kardar–Parisi–Zhang (KPZ)** equation.
- ▶ The earliest proof of KPZ convergence is due to Bertini & Giacomin '97, who showed it for the **weakly asymmetric simple exclusion process (WASEP)**.
- ▶ Since then, many other one-dimensional growing surfaces have been shown to have a KPZ scaling limit.

# The KPZ equation

- ▶ The KPZ equation was introduced by Kardar, Parisi & Zhang '86 to describe the growth of a generic randomly growing surface.
- ▶ Formally, the KPZ equation is

$$\partial_t f = \beta \Delta f + \gamma |\nabla f|^2 + \kappa \xi,$$

where  $\xi$  is a random field known as **space-time white noise**, and  $\beta$ ,  $\gamma$  and  $\kappa$  are real-valued parameters.

- ▶ A foundational challenge is to give a rigorous meaning to the KPZ equation.
- ▶ In  $d = 1$ , there are now many approaches to solving this problem, such as the Cole–Hopf solution (Bertini & Giacomin '97), regularity structures (Hairer '13 '14), paracontrolled distributions (Gubinelli, Imkeller & Perkowski '15), energy solutions (Gonçalves and Jara '12 '14) and renormalization group (Kupiainen and Marozzi '17).



- ▶ KPZ scaling limit has been established for many 1D surfaces.
- ▶ Some examples:
  - ▶ Directed polymers in the intermediate disorder regime (Amir, Corwin & Quastel '11, Alberts, Khanin & Quastel '14),
  - ▶ Polynuclear growth (Prähofer & Spohn '02).
  - ▶ Weakly asymmetric exclusion processes (Sasamoto & Spohn '10, Dembo & Tsai '16).
  - ▶ Log-gamma polymers (Borodin, Corwin & Remenik '13).
  - ▶ Macdonald processes (Borodin & Corwin '14).
- ▶ Many of these results are based on exact formulas derived in prior work, such as Mueller '91, Baik, Deift & Johansson '99, Johansson '00, Corwin, O'Connell, Seppäläinen and Zygouras '14, Sasamoto & Spohn '10, Tracy & Widom '08 '09.
- ▶ Intense recent activity on related objects, such as the KPZ fixed point (Matetski, Quastel & Remenik '17), the KPZ line ensemble (Corwin & Hammond '16), and the Brownian landscape (Dauvergne, Ortmann & Virág '17).

- ▶ Very few exact formulas are available (Prähofer & Spohn '97, Borodin & Ferrari '14, Toninelli '18, Chhita & Toninelli '19).
- ▶ There has been a limited amount of recent progress in making sense of the KPZ equation in  $d \geq 2$  (Magen & Unterberger '18, Comets, Cosco & Mukherjee '19 '20, Chatterjee & Dunlap '20, Caravenna, Sun & Zygouras '20, Dunlap, Gu, Ryzhik & Zeitouni '20, Gu '20, Lygkonis & Zygouras '20).
- ▶ However, no nontrivial discrete process has been shown to have a KPZ scaling limit in  $d \geq 2$ .

# KPZ universality

- ▶ KPZ universality is the ill-posed conjecture that a large class of discrete processes should have a KPZ scaling limit.
- ▶ As we saw, this has been established almost exclusively for integrable (i.e., exactly solvable) models in  $d = 1$ .
- ▶ A certain kind of universality of the KPZ equation in  $d = 1$  was established by Hairer & Quastel '18, who showed that if the  $|\nabla f|^2$  term is replaced by a polynomial in  $\nabla f$ , and the white noise is replaced by the mollification of itself, the  $|\nabla f|^2$  term reappears in the limit as the mollification is removed.
- ▶ This result was later improved by Gubinelli & Perkowski '16, Hairer & Xu '19, and Yang '20a.
- ▶ The convergence of some non-integrable models to 1D KPZ was established by Dembo & Tsai '16, Yang '20b, '20c, and Quastel & Sarkar '20.
- ▶ In  $d \geq 2$ , there are no mathematical results about KPZ universality.

# An attempt at KPZ universality without integrability

- ▶ The rest of the talk is about a fledgling attempt towards building a 'general theory' of surface growth in the absence of integrability, introduced in the following preprints that I put up on arXiv in February and March of 2021:
  - ▶ **Universality of deterministic KPZ.** arXiv:2102.13131
  - ▶ **Superconcentration in surface growth.** arXiv:2103.09199
- ▶ The approach is still in its beginning stages, and the results obtained in the above preprints are far less definitive than the available one-dimensional results. But it introduces a new way of looking at the problem which may open up new possibilities.

# A general growth mechanism

- ▶ Let  $e_1, \dots, e_d$  be the standard basis vectors of  $\mathbb{R}^d$ .
- ▶ Let  $A := \{0, \pm e_1, \pm e_2, \dots, \pm e_d\}$  and  $B := A \setminus \{0\}$ . The sets  $A$  and  $B$  will be fixed throughout the rest of this talk.
- ▶ Let  $\phi : \mathbb{R}^A \times \mathbb{R} \rightarrow \mathbb{R}$  be a function.
- ▶ Let  $z = \{z_{t,x} : t \in \mathbb{Z}_{>0}, x \in \mathbb{Z}^d\}$  be a collection of i.i.d. random variables.
- ▶ We will say that the evolution of a  $d$ -dimensional growing random surface  $f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d \rightarrow \mathbb{R}$  is driven by the function  $\phi$  and the noise field  $z$  if for each  $t \in \mathbb{Z}_{\geq 0}$  and  $x \in \mathbb{Z}^d$ ,

$$f(t+1, x) = \phi((f(t, x+a))_{a \in A}, z_{t+1, x}).$$

- ▶ In other words, we are allowing the growth of the height function at a point to depend arbitrarily on the current heights at that point and its neighboring points, plus a random component.
- ▶ One deficiency of this framework is that it does not include continuous-time models.

# Assumptions about the driving function

- ▶ Throughout, we will assume that the driving function  $\phi$  has the following natural properties:
  - ▶ **Equivariance under constant shifts.** For  $u \in \mathbb{R}^A$  and  $c \in \mathbb{R}$ , let  $u + c$  denote the vector obtained by adding  $c$  to each coordinate of  $u$ . We assume that  $\phi(u + c, z) = \phi(u, z) + c$  for each  $u, c$  and  $z$ . (In other words, if the whole surface at time  $t$  is raised by a fixed constant  $c$ , then the surface at time  $t + 1$  is also raised by the same constant amount.)
  - ▶ **Monotonicity.** We assume that  $\phi$  is monotone increasing in the first argument. That is, if  $u$  dominates  $v$  in each coordinate, then  $\phi(u, z) \geq \phi(v, z)$  for any  $z$ . (Thus, a surface that is higher than another surface everywhere at time  $t$  will continue to remain so at time  $t + 1$ .)
- ▶ The above assumptions seem quite natural from a physical point of view, and are satisfied by many of the standard discrete-time models.
- ▶ The interesting thing is that these assumptions also give rise to useful random walk representations.

- ▶ To start analyzing the general growth mechanism, let us begin with the assumption that the growth is fully deterministic.
- ▶ That is,

$$f(t+1, x) = \phi((f(t, x+a))_{a \in A})$$

for a driving function  $\phi$  that is equivariant under constant shifts and monotone. (Recall that  $A = \{0, \pm e_1, \dots, \pm e_d\}$ .)

- ▶ **Main question:** Can we say something about a **scaling limit of  $f$**  under a proper formulation?

## Two additional assumptions

- ▶ In addition to equivariance under constant shifts and monotonicity, we make the following two extra assumptions about  $\phi$ :
  - ▶ **Invariance under lattice symmetries.** We assume that  $\phi$  is invariant under the following set of symmetries. Let  $u \in \mathbb{R}^A$ , and let  $v$  be obtained by swapping  $u_{e_i}$  with  $u_{e_j}$  and  $u_{-e_i}$  with  $u_{-e_j}$  for some  $i$  and  $j$ . Then we assume that  $\phi(u) = \phi(v)$ . Also, if  $v$  is obtained by swapping  $u_{e_i}$  with  $u_{-e_i}$  for some  $i$ , we assume that  $\phi(u) = \phi(v)$ . (This means that the surface does not prefer to grow differently in different lattice directions.)
  - ▶ **Twice continuous differentiability.** We assume that  $\phi$  is twice continuously differentiable. (We will see later that this regularity condition is important.)



- ▶ Let  $\phi : \mathbb{R}^A \rightarrow \mathbb{R}$  be a driving function that is equivariant under constant shifts, monotone, invariant under lattice symmetries and twice continuously differentiable.
- ▶ Let  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  be a Lipschitz function.
- ▶ For each  $\varepsilon > 0$ , define  $g_\varepsilon : \mathbb{Z}^d \rightarrow \mathbb{R}$  as

$$g_\varepsilon(x) := g(\varepsilon x).$$

- ▶ Let  $f_\varepsilon : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d \rightarrow \mathbb{R}$  be the function obtained using the evolution

$$f_\varepsilon(t+1, x) = \phi((f_\varepsilon(t, x+a))_{a \in A})$$

with initial condition  $f_\varepsilon(0, \cdot) = g_\varepsilon(\cdot)$ .

- ▶ **Objective:** Understand the behavior of  $f_\varepsilon$  as  $\varepsilon \rightarrow 0$ , after scaling time and space appropriately, depending on  $\varepsilon$ .

# Parabolic scaling of time and space

- ▶ For  $t \in \mathbb{R}$ , let  $[t]$  denote the greatest integer  $\leq t$ .
- ▶ For  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , let  $[x]$  denote the vector  $([x_1], \dots, [x_d])$ .
- ▶ For  $t \in \mathbb{R}_{\geq 0}$ ,  $x \in \mathbb{R}^d$ , and  $\varepsilon > 0$ , let

$$t_\varepsilon := [\varepsilon^{-2}t], \quad x_\varepsilon := [\varepsilon^{-1}x].$$

- ▶ Finally, define  $f^{(\varepsilon)} : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \rightarrow \mathbb{R}$  as

$$f^{(\varepsilon)}(t, x) := f_\varepsilon(t_\varepsilon, x_\varepsilon) - t_\varepsilon \phi(0).$$

- ▶ We will investigate the limit of  $f^{(\varepsilon)}$  as  $\varepsilon \rightarrow 0$ .

## Two important quantities

- ▶ Recall:  $A = \{0, \pm e_1, \dots, \pm e_d\}$ ,  $B = \{\pm e_1, \dots, \pm e_d\}$ .
- ▶ For  $a \in A$ , let  $\partial_a \phi(u)$  denote the partial derivative of  $\phi$  with respect to  $u_a$ .
- ▶ By the invariance under lattice symmetries,  $\partial_b \phi(0)$  is the same for all  $b \in B$ . Let this quantity be denoted by  $\beta$ .
- ▶ Similarly, the second order derivatives  $\partial_b^2 \phi(0)$  and  $\partial_b \partial_{-b} \phi(0)$  do not depend on the choice  $b \in B$ . Define

$$\gamma := \partial_b^2 \phi(0) - \partial_b \partial_{-b} \phi(0).$$

- ▶ Note that  $\beta$  is nonnegative due to the monotonicity of  $\phi$ .

## Theorem (Chatterjee '21a)

As  $\varepsilon \rightarrow 0$ ,  $f^{(\varepsilon)}$  converges pointwise to the unique solution of the *deterministic KPZ equation*

$$\partial_t f = \beta \Delta f + \gamma |\nabla f|^2$$

with initial condition  $f(0, \cdot) = g(\cdot)$ .

# Explicit form of the solution

- ▶ There are three cases.
- ▶ First, suppose that  $\beta$  and  $\gamma$  are both nonzero.
- ▶ Then for any  $t > 0$  and any  $x \in \mathbb{R}^d$ ,

$$f(t, x) = \frac{\beta}{\gamma} \log \int_{\mathbb{R}^d} K(t, x - y) e^{\gamma g(y)/\beta} dy,$$

where  $K(t, x) = (4\pi\beta t)^{-d/2} e^{-|x|^2/4\beta t}$ . This is the **Cole–Hopf formula** for the solution of the deterministic KPZ equation.

- ▶ Next, suppose that  $\beta \neq 0$  and  $\gamma = 0$ . Then

$$f(t, x) = \int_{\mathbb{R}^d} K(t, x - y) g(y) dy.$$

This is just the convolution formula for the solution of the heat equation.

- ▶ Finally, if  $\beta = 0$ , then  $\gamma$  must also be zero, and in this case  $f(t, x) = g(x)$  for all  $t$  and  $x$ .

## Some examples

- ▶ Recall:  $A = \{0, \pm e_1, \dots, \pm e_d\}$ ,  $B = \{\pm e_1, \dots, \pm e_d\}$ , and  $\phi$  is a function from  $\mathbb{R}^A$  into  $\mathbb{R}$ .
- ▶ **Edwards–Wilkinson model:**  $\phi(u) = \frac{1}{2d} \sum_{b \in B} u_b$ .
- ▶ **Directed polymer model at inverse temperature  $\theta$ :**  
 $\phi(u) = \frac{1}{\theta} \log \sum_{b \in B} e^{\theta u_b}$ .
- ▶ In the above models, the discrete evolution can be explicitly solved. But that is not the case in the following.
- ▶ Any function of the form

$$\phi(u) = u_0 + \frac{1}{2d} \sum_{b \in B} q(u_b - u_0),$$

where  $q \in C^2$  and  $0 \leq q' \leq 1$  everywhere, satisfies the required conditions.

## Further examples

- ▶ More generally, the conditions are satisfied by any function of the form  $\phi(u) = u_0 + cF((u_b - u_0)_{b \in B})$ , where  $F$  is a nondecreasing symmetric  $C^2$  function on  $\mathbb{R}^B$  with bounded derivatives, and  $c$  is a sufficiently small constant. (This is sort of a generalization of the [Hairer–Quastel universality result](#) to higher dimensions, but without noise.)
- ▶ Another class of examples arise as deterministic versions of [Glauber dynamics for gradient Gibbs measures with convex potentials](#):

$$\phi(u) = \frac{\int_{-\infty}^{\infty} t \exp(-\sum_{b \in B} V(u_b - t)) dt}{\int_{-\infty}^{\infty} \exp(-\sum_{b \in B} V(u_b - t)) dt},$$

where  $V$  is convex, even, twice continuously differentiable, and sufficiently nicely behaved to allow derivatives with respect to  $u$  to be moved inside the above integrals. (For details, see the preprint.)

## A remark about the $C^2$ assumption

- ▶ The  $C^2$  assumption on  $\phi$  is quite restrictive. There are many natural examples that satisfy the other three assumptions but are not  $C^2$ .
- ▶ Some examples:

$$\phi(u) = \max_{b \in B} u_b.$$

$$\phi(u) = \frac{1}{2}(\max_{b \in B} u_b + \min_{b \in B} u_b).$$

$$\phi(u) = \text{median}\{u_b : b \in B\}.$$

- ▶ The first one comes from [last-passage percolation](#) (investigated by Krug & Spohn '88 in  $d = 1$ ), the second one from a deterministic version of the [restricted solid-on-solid \(RSOS\)](#) growth model, and the third one comes from a deterministic version of the Glauber dynamics for the [solid-on-solid \(SOS\)](#) model.



## Forthcoming work

- ▶ In forthcoming work with P. Souganidis, we are able to remove the  $C^2$  assumption.
- ▶ If the  $C^2$  assumption does not hold, the limit is not necessarily deterministic KPZ. We are getting various unfamiliar nonlinear PDEs as limits.
- ▶ The proof of the universality theorem in Chatterjee '21a involves a lengthy bare-hands argument. The proofs in the forthcoming paper with Souganidis use the Crandall–Lions theory of viscosity solutions and a general method of Barles & Souganidis '91. This allows considerable simplification of the proofs.

## Part II: Noisy growth

- ▶ Recall the general growth mechanism

$$f(t+1, x) = \phi((f(t, x+a))_{a \in A}, z_{t+1, x}),$$

where the  $z$ 's are i.i.d. random variables.

- ▶ We will assume that  $\phi$  is equivariant under constant shifts and monotone in the first argument, as before.
- ▶ But we will **not** assume invariance under lattice symmetries and twice continuous differentiability.
- ▶ The main reason for dropping the last two assumptions is that our objective is much more humble than before: We will simply try to get some upper bounds on the fluctuations of the height function, instead of trying to prove convergence to KPZ (which seems to be fairly out reach at this moment).

# Superconcentration of the height function

- ▶ Recall the **random deposition model**:  
$$f(t+1, x) = f(t, x) + z_{t+1, x}.$$
- ▶ Here  $\text{Var}(f(t, x))$  grows linearly in  $t$ .
- ▶ But this seems to be the exception rather than the rule.
- ▶ In most models of interest, where the growth at a point is influenced by the heights at neighboring points, it is believed that  $\text{Var}(f(t, x))$  grows sublinearly (i.e., like  $o(t)$ ).
- ▶ This is sometimes called **superconcentration**.
- ▶ Roughly speaking, a random variable is superconcentrated if its order of fluctuations is smaller than what is obtainable by the classical methods of concentration of measure.
- ▶ It turns out that superconcentration has important implications for the structure of a random field — it is equivalent to the phenomenon of chaos and implies the existence of multiple valleys. (See the monograph of Chatterjee '14 for details.)

## Some conjectures

- ▶ In  $d = 1$ , models in the 'KPZ universality class' are supposed to have  $\text{Var}(f(t, x))$  growing like  $t^{2/3}$ . This has been proved for integrable models, but remains open for others.
- ▶ In  $d = 2$ , it was conjectured by Kim & Kosterlitz '89 that  $\text{Var}(f(t, x)) \sim Ct^{1/2}$  for models in the KPZ class.
- ▶ In some fascinating large scale simulations in recent years by Parisi and collaborators, the validity of this conjecture has been questioned. They claim that  $\text{Var}(f(t, x))$  grows like  $t^\alpha$  for some  $\alpha$  slightly less than  $1/2$  (roughly, like 0.48).
- ▶ Except for integrable one-dimensional models, all of the above are currently far out of the reach of mathematical techniques.
- ▶ In  $d \geq 2$ , even just proving superconcentration has been possible only for the last-passage percolation model (by Chatterjee '08 and Graham '12, based on technology developed by Kahn, Kalai & Linial '88, Talagrand '94 and Benjamini, Kalai & Schramm '03).

## Two additional assumptions

- ▶ I will now present an attempt to understand superconcentration in growing surfaces in some generality.
- ▶ Recall:

$$f(t+1, x) = \phi((f(t, x+a))_{a \in A}, z_{t+1, x}),$$

where the  $z$ 's are i.i.d. random variables, and  $\phi$  is equivariant under constant shifts and monotone in the first argument.

- ▶ **Gaussian noise:** Since any random variable can be expressed as a function of a standard Gaussian random variable, and  $\phi$  is an arbitrary function, we will assume without loss of generality that the  $z$ 's are i.i.d. standard Gaussian random variables.
- ▶ **Lipschitzness in the noise variable:** We will assume that  $\phi$  is uniformly Lipschitz in the second argument, with Lipschitz constant  $L$ . That is,  $|\phi(u, z) - \phi(u, z')| \leq L|z - z'|$  for all  $u$ ,  $z$  and  $z'$ . Fortunately, this technical assumption is satisfied in many examples.

## First step: Variance can grow at most linearly

The following result shows that  $\text{Var}(f(t, x))$  grows at most linearly in  $t$  in our general setup, together with an exponentially decaying tail bound. The random deposition model shows that the bound is sharp.

### Theorem (Chatterjee '21b)

*For all  $t \geq 1$  and  $x \in \mathbb{Z}^d$ ,  $\text{Var}(f(t, x)) \leq L^2 t$ . Moreover, for all  $r \geq 0$ ,  $\mathbb{P}(|f(t, x) - \mathbb{E}(f(t, x))| \geq r) \leq 2e^{-r^2/2L^2 t}$ .*

The proof uses a random walk representation that I will show if time permits.

# Subroughness and superconcentration

- ▶ Henceforth, let us assume that  $f(0, \cdot) \equiv 0$ .
- ▶ We will say that the surface  $f$  is **superconcentrated** if

$$\lim_{t \rightarrow \infty} \frac{\text{Var}(f(t, x))}{t} = 0.$$

(Note that the term on the left does not depend on  $x$  due to the assumption that  $f(0, \cdot) \equiv 0$ .)

- ▶ We will say that the surface is **subrough** if there exist two distinct points  $x, y \in \mathbb{Z}^d$  such that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[(f(t, x) - f(t, y))^2]}{t} = 0.$$

- ▶ Lastly, we will say that the surface is **completely subrough** if the above equality holds for any two distinct points  $x$  and  $y$ .

## Theorem (Chatterjee '21b)

*For the surface  $f$ , superconcentration, subroughness and complete subroughness are equivalent.*

- ▶ How can this theorem be used?
- ▶ Often, it may be easier to show that at time  $t$ , the heights at neighboring points differ by  $o(\sqrt{t})$ , rather than showing that the order of fluctuations of the height is  $o(\sqrt{t})$ . By the theorem, these two are equivalent.



## Quantitative version

- ▶ For each  $t \geq 1$ , define

$$\alpha_t := \frac{\text{Var}(f(t, x))}{L^2 t}.$$

- ▶ Next, for any  $b \in \mathbb{Z}^d$  and  $t \geq 1$ , define

$$\beta_{b,t} := \frac{\mathbb{E}[(f(t, x) - f(t, x + b))^2]}{4L^2 t}.$$

- ▶ Note that above quantities have no dependence on  $x$  since  $f(0, \cdot) \equiv 0$ .

### Theorem (Chatterjee '21b)

*There is a constant  $C(d)$  depending only on  $d$  such that for any  $b \neq 0$  and  $t \geq 1$ ,  $\beta_{b,t} \leq \alpha_t \leq C(d)/|\log \beta_{b,t}|$ .*

(Thus,  $\lim_{t \rightarrow \infty} \beta_{b,t} = 0$  for some  $b$  if and only if  $\lim_{t \rightarrow \infty} \alpha_t = 0$  if and only if  $\lim_{t \rightarrow \infty} \beta_{b,t} = 0$  for all  $b$ . This proves the equivalence theorem.)

## A variant of the RSOS model

- ▶ The restricted solid-on-solid (RSOS) model is a popular toy model of surface growth introduced by Kim & Kosterlitz '89.
- ▶ There are many variants of this model, all built on one basic principle: The growing surface has to satisfy, at all times, that the differences between the heights at neighboring points are uniformly bounded by some given constant (usually 1).
- ▶ Essentially no nontrivial rigorous results are known above dimension one. In  $d = 1$ , exclusion processes are variants of RSOS. Some of those are integrable.
- ▶ We will work with the following variant in  $\mathbb{Z}^d$ .
- ▶ Consider  $\mathbb{Z}^d$  as a bipartite graph, splitting the set of vertices into 'even' and 'odd' vertices.
- ▶ Alternately update the heights at even and odd vertices, choosing independently and uniformly among all values that maintain the constraint that the differences between the heights at neighboring points are uniformly bounded by 1.

# Superconcentration in our variant of the RSOS model

Using the quantitative equivalence theorem, we get the following bound, which proves superconcentration of the height function of the RSOS model in any dimension.

## Theorem (Chatterjee '21b)

*Let  $f$  be the height function in the variant of the RSOS model defined above. There is a constant  $C(d)$ , depending only on the dimension  $d$ , such that for any  $t \geq 2$  and  $x \in \mathbb{Z}^d$ ,*

$$\text{Var}(f(t, x)) \leq C(d)t / \log t.$$

# Ballistic deposition

- ▶ Ballistic deposition is a popular model of surface growth introduced by Vold '59.
- ▶ In the continuous-time version we have a height function  $f : \mathbb{R}_{\geq 0} \times \mathbb{Z}^d \rightarrow \mathbb{R}$ , which grows as follows.
- ▶ There is an independent Poisson clock at each  $x$ .
- ▶ When the clock at  $x$  rings, a brick of height 1 drops on the surface at location  $x$  'from infinity', as in a game of Tetris.
- ▶ As the brick descends, it can either attach itself to the surface at  $x$ , thereby increasing the height at  $x$  by 1, or it can get 'stuck' to the side of one of the neighboring columns if that happens before it reaches the surface.
- ▶ Thus, if the clock at  $x$  rings at time  $t$ , then the height  $f(t, x)$  instantly increases to  $\max\{f(t, x) + 1, \max_{b \in B} f(t, x + b)\}$ .

# Known results

- ▶ Physicists say that this model is in the KPZ class.
- ▶ On the mathematical side, the only results we know are the following.
- ▶ A strong law of large numbers for the height function was proved by Seppäläinen '00.
- ▶ A central limit theorem for the total height in a large region at a finite time  $t$  was proved by Penrose & Yukich '02.
- ▶ Penrose '08 proved that the variance of  $f(t, x)$  grows at least like  $\log t$  when  $d = 1$ .
- ▶ In particular, **no nontrivial upper bound on the variance is known.**

# A variant of ballistic deposition

- ▶ Instead of bricks falling at random times, our discrete-time model will update the heights at all sites simultaneously.
- ▶ To insert randomness, we will make the brick heights random. For definiteness, let us take the brick heights to be i.i.d.  $\text{Uniform}[0, 1]$  random variables.
- ▶ In other words, the height function  $f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d \rightarrow \mathbb{R}$  satisfies

$$f(t+1, x) = \max\{f(t, x) + v_{t+1, x}, \max_{b \in B} f(t, x + b)\},$$

where the  $v$ 's are i.i.d.  $\text{Uniform}[0, 1]$  random variables.

- ▶ As usual, we start from  $f(0, \cdot) \equiv 0$ .

# Superconcentration and subroughness in our variant of ballistic deposition

## Theorem (Chatterjee '21b)

*In our variant of ballistic deposition,*

$$\mathbb{E}[(f(t, x) - f(t, y))^2] \leq C(d)t^{3/4} \log t$$

*for any two neighboring points  $x$  and  $y$ . Consequently, by the quantitative equivalence theorem,  $\text{Var}(f(t, x)) \leq C(d)t/\log t$ .*

# First step in the proof: A random walk representation

- ▶ As a consequence of equivariance under constant shifts and monotonicity, it follows that for any  $(u, z) \in \mathbb{R}^A \times \mathbb{R}$ ,  $\partial_a \phi(u, z) \geq 0$  for each  $a \in A$ , and  $\sum_{a \in A} \partial_a \phi(u, z) = 1$ .
- ▶ Given a realization of  $f$ , this allows us to define the following random walk.
- ▶ The walk starts at a point  $x \in \mathbb{Z}^d$  at some time  $t \geq 1$ , and goes backwards in time, until reaching time 0.
- ▶ If the walk is at location  $y \in \mathbb{Z}^d$  at time  $s \geq 1$ , then at time  $s - 1$  it moves to  $y + a$  with probability  $\partial_a \phi((f(s - 1, y + a))_{a \in A}, z_{s, y})$ , for  $a \in A$ .
- ▶ By the above observation, these numbers are nonnegative and sum to 1 when summed over  $a \in A$ . Therefore, this describes a legitimate random walk on  $\mathbb{Z}^d$ , moving backwards in time.
- ▶ In some sense, this is a generalization of **geodesics** in the last-passage percolation model.



## Lemma (Chatterjee '21b)

Take any  $1 \leq s \leq t$  and  $x, y \in \mathbb{Z}^d$ . Given a realization of  $f$ , let  $\{S_r\}_{0 \leq r \leq t}$  be the backwards random walk defined above, started at  $x$  at time  $t$ . Then

$$\frac{\partial}{\partial z_{s,y}} f(t, x) = \mathbb{P}(S_s = y) \partial_z \phi((f(s-1, y+a))_{a \in A}, z_{s,y}).$$

- ▶ This lemma gives us information about the influence of the noise variables on the height function, which is the key input for concentration of measure. This gives the required starting point for carrying out the rest of the proofs.
- ▶ The lemma is a consequence of only the equivariance and monotonicity properties. The Gaussian nature of the noise and the Lipschitzness of  $\phi$  in the noise variable are used for applying concentration inequalities.

# Summary

- ▶ In two recent preprints, I have attempted some small steps towards surface growth beyond integrable models.
- ▶ In the first preprint, it is shown that a set of natural assumptions on a deterministic growth mechanism leads to a **deterministic KPZ scaling limit**. Forthcoming work with Souganidis generalizes this to obtain new scaling limits.
- ▶ The second preprint investigates the phenomenon of **superconcentration** in growing random surfaces.
- ▶ Again under a set of natural assumptions on the growth mechanism, it is shown that superconcentration happens if and only if the surface has a property that I call **subroughness**.
- ▶ Subroughness turns out to be easier to prove than superconcentration. Following this route, it is shown that superconcentration happens in variants of two popular models of surface growth (RSOS and ballistic deposition).
- ▶ **Many open questions!**

*Thanks for listening!*