# Superconcentration

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- Examples: directed polymers, last passage percolation, spin glasses, the discrete Gaussian free field, random matrix eigenvectors, fitness models of evolutionary biology, etc.
- We will illustrate the theory through a single example in this talk: the Sherrington-Kirkpatrick model of spin glasses.

# The Sherrington-Kirkpatrick model

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- $H_N$ is the Hamiltonian (or energy function) in the Sherrington-Kirkpatrick model of spin glasses. Note that it is a random function on $\{-1, 1\}^N$.

# Some history

- Introduced by Sherrington and Kirkpatrick in 1975; first breakthrough by Thouless-Anderson-Palmer in 1977; revolutionary development of the broken replica method by Parisi and Mézard in late 70's and early 80's; first rigorous analysis of the high temperature phase by Aizenman-Lebowitz-Ruelle in 1987, and later by Fröhlich-Zegarlinski, Comets-Neveu; high temperature phase under nonzero external field studied by Talagrand and Shcherbina in the late 90's; great advancement in the rigorous understanding of the low temperature phase due to breakthroughs of Guerra, Toninelli, Talagrand and Panchenko between 2001 and 2008. Most significant breakthrough: Proof of the Parisi formula by Talagrand in 2003.

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- Still, many mysteries.

# The overlap

- For $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2 \in \{-1, 1\}^N$, define

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- In the spin glass literature, the quantity $R_{1,2}$ is called the overlap between the configurations $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$.

# The multiple valley question

- Recall: The energy of a state $\boldsymbol{\sigma}$ is defined as

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- Is it true that with high probability, there exists a large number of mutually nearly-orthogonal states that all have nearly minimum-energy?
- Origin: folklore in the physics literature. See Chapter III in Mézard-Parisi-Virasoro '87.
- No rigorous formulation or results till now.

# A counterexample

- To realize the non-triviality of the question, consider a slightly different Gaussian field $Y_N$ on $\{-1,1\}^N$, defined as

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  we see that if $\boldsymbol{\sigma}$ is another configuration that is near-minimal for $Y_N$, then $\boldsymbol{\sigma}$ must agree with $\hat{\boldsymbol{\sigma}}$ at nearly all coordinates.

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- Thus, the field $Y_N$ does not have multiple valleys.
- This is true in spite of $Y_N(\boldsymbol{\sigma})$ and $Y_N(\boldsymbol{\sigma}')$ being nearly independent for most $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$.

# Weak resolution of the multiple valley conjecture

Recall: $R_{\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2} = \frac{1}{N}\sum \sigma_i^1\sigma_i^2$, $H_N(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}}\sum g_{ij}\sigma_i\sigma_j$.

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## Theorem (C. '09)

*There are constants $r_N \to \infty$, $\gamma_N \to 0$, $\epsilon_N \to 0$, and $\delta_N \to 0$ such that with probability at least $1-\gamma_N$, there is a set $A \subseteq \{-1,1\}^N$ satisfying*

(a) $|A| \geq r_N$,

(b) $R^2_{\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2} \leq \epsilon_N$ *for all* $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2 \in A$, $\boldsymbol{\sigma}^1 \neq \boldsymbol{\sigma}^2$, *and*

(c) *For all* $\boldsymbol{\sigma} \in A$,

$$\left|\frac{H_N(\boldsymbol{\sigma})}{\min_{\boldsymbol{\sigma}' \in \{-1,1\}^N} H_N(\boldsymbol{\sigma}')} - 1\right| \leq \delta_N.$$

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*Quantitatively, we can take $r_N = (\log N)^{1/8}$, $\delta_N = (\log N)^{-1/8}$, $\epsilon_N = e^{-(\log N)^{1/8}}$ and $\gamma_N = C(\log N)^{-1/12}$, where $C$ is an absolute constant. However these are not necessarily the best choices.*

# The Gibbs measure of the S-K model

- The S-K model at inverse temperature $\beta \geq 0$ defines a probability measure $G_N$ on $\{-1,1\}^N$ through the formula

$$G_N(\{\boldsymbol{\sigma}\}) := Z(\beta)^{-1} e^{-\beta H_N(\boldsymbol{\sigma})}, \tag{1}$$

where $Z(\beta)$ is the normalizing constant. The measure $G_N$ is called the Gibbs measure. Recall that

$$H_N(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j.$$

# Disorder chaos in the S-K model

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- Suppose we choose $\boldsymbol{\sigma}^2$ from a new Gibbs measure $G'_N$, based on a new Hamiltonian $H'_N$ obtained by applying a small perturbation to $H_N$.
- The conjecture of disorder chaos (Fisher-Huse '85, Bray-Moore '87) states that in this case, $R_{1,2} \simeq 0$. (Any $\beta$.)

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- Again, no rigorous formulation or proof in the past. Seems related to noise-sensitivity, although we do not understand the exact connection.

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**Theorem (C. '09)**

*Let $\boldsymbol{\sigma}^1$ be chosen from the original Gibbs measure and $\boldsymbol{\sigma}^2$ is chosen from the $p$-perturbed measure. Then*

$$\mathbb{E}(R_{1,2}^2) \le \frac{C\beta}{p \log N},$$

*where $C$ is an absolute constant and the expectation is taken over all randomness.*

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- Since $\beta \to \infty$, $\boldsymbol{\sigma}^1$ nearly minimizes $H_N$ and $\boldsymbol{\sigma}^2$ nearly minimizes $H'_N$.
- But, since $p \to 0$, $H_N \approx H'_N$.
- Thus, $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$ both nearly minimize $H_N$.
- This procedure finds two states that have nearly minimal energy and are nearly orthogonal. Repeating this procedure, we find many such states.

# Superconcentration

- The free energy of the S-K model at inverse temperature $\beta$ is:

$$F_N(\beta) := \frac{1}{\beta} \log \sum_{\boldsymbol{\sigma}} e^{-\beta H_N(\boldsymbol{\sigma})}.$$

Recall: $H_N(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}} \sum g_{ij} \sigma_i \sigma_j$.

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- Standard concentration of Lipschitz functions under the Gaussian measure implies that $\mathrm{Var}(F_N(\beta)) \le N$.
- It is known that $\mathrm{Var}(F_N(\beta)) \le \mathit{const}$. if $\beta < 1$. (Aizenman-Lebowitz-Ruelle, Fröhlich-Zegarlinski, Comets-Neveu, Talagrand, Shcherbina, etc.)

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- Standard concentration of Lipschitz functions under the Gaussian measure implies that $\mathrm{Var}(F_N(\beta)) \le N$.
- It is known that $\mathrm{Var}(F_N(\beta)) \le \textit{const}.$ if $\beta < 1$. (Aizenman-Lebowitz-Ruelle, Fröhlich-Zegarlinski, Comets-Neveu, Talagrand, Shcherbina, etc.)
- However, $\mathrm{Var}(F_N(\beta)) \le N$ was the best available bound for $\beta > 1$ till now.

# Superconcentration

- The free energy of the S-K model at inverse temperature $\beta$ is:

$$F_N(\beta) := \frac{1}{\beta} \log \sum_{\boldsymbol{\sigma}} e^{-\beta H_N(\boldsymbol{\sigma})}.$$

  Recall: $H_N(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}} \sum g_{ij}\sigma_i\sigma_j$.
- As a function of the Gaussian random vector $(g_{ij})_{1\le i<j\le N}$, $F_N(\beta)$ is Lipschitz with Lipschitz constant $N$.
- Standard concentration of Lipschitz functions under the Gaussian measure implies that $\mathrm{Var}(F_N(\beta)) \le N$.
- It is known that $\mathrm{Var}(F_N(\beta)) \le \mathit{const}.$ if $\beta < 1$. (Aizenman-Lebowitz-Ruelle, Fröhlich-Zegarlinski, Comets-Neveu, Talagrand, Shcherbina, etc.)
- However, $\mathrm{Var}(F_N(\beta)) \le N$ was the best available bound for $\beta > 1$ till now.
- We claim that for any $\beta$, $F_N(\beta)$ is superconcentrated, meaning that $\mathrm{Var}(F_N(\beta)) = o(N)$.

# Superconcentration of the free energy

**Theorem (C. '09)**

*Let $F_N(\beta)$ be the free energy of the S-K model. Then for any $\beta$,*

$$\mathrm{Var}(F_N(\beta)) \leq \frac{CN \log(1 + C\beta)}{\log N},$$

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- Has been called 'sublinear variance' or 'submean variance' before. 'Superconcentration' is probably more evocative.
- In [Chatterjee '08], it was shown that superconcentration is equivalent to chaos and multiple valleys in a general setting. Therefore, it is more than just a curious phenomenon.

# How to prove discrete chaos using superconcentration

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$$\mathbb{E}(R_{1,2}^2) \le C\frac{\mathrm{Var}(F_N(\beta))}{pN} + C\beta N^{-1/2},$$

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# Continuous perturbation

- Replace $g_{ij}$ by $ag_{ij} + bg'_{ij}$, where $(g'_{ij})$ is another set of indepenent standard Gaussian random variables and $a^2 + b^2 = 1$.

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- Perturbing the couplings up to time $t$ corresponds to running an Ornstein-Uhlenbeck flow at each coupling for time $t$, with initial value $g_{ij}$.
- The new Gibbs measure will be called the $t$-perturbed measure.

# Chaos under continuous perturbation

**Theorem (C. '09)**

*Let $\boldsymbol{\sigma}^1$ be chosen from the original Gibbs measure and $\boldsymbol{\sigma}^2$ be chosen from the t-perturbed measure. Then there is an absolute constant $C$ such that for any positive integer $k$,*

$$\mathbb{E}(R_{1,2}^{2k}) \le (Ck)^k N^{-k \min\{1,\, t/C \log(1+C\beta)\}}.$$

# How to prove superconcentration $\iff$ continuous chaos

## Theorem (C. '09)

*Let $\phi(t)$ denote $\mathbb{E}(R_{1,2}^2)$ when $\boldsymbol{\sigma}^2$ is drawn from the t-perturbed measure. Let $F_N(\beta)$ be the free energy. Then*

$$\mathrm{Var}(F_N(\beta)) = N \int_0^\infty e^{-t}\phi(t)dt.$$

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- We will also show that $\phi$ is a nonnegative and decreasing function. This proves the converse implication.
- By our chaos theorem for continuous perturbation, $\phi(t) \le CN^{-\min\{1,t/C(\beta)\}}$. This shows that

$$\mathrm{Var}(F_N(\beta)) \le \frac{C(\beta)N}{\log N}.$$

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- Continuous chaos $\iff$ superconcentration of free energy $\implies$ discrete chaos $\implies$ multiple valleys.
- Remains to prove: continuous chaos.

# Proof of chaos under continuous perturbation - 1

- Suppose $\boldsymbol{\sigma}^1$ is drawn from the Gibbs measure and the $\boldsymbol{\sigma}^2$ from the $t$-perturbed measure. Recall: $R_{1,2} = \frac{1}{N}\sum \sigma_i^1 \sigma_i^2$. Let

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- By repeated applications of differentiation and Gaussian integration-by-parts, we show that $(-1)^j\phi_k^{(j)}(t) \ge 0$ for all $t$ and $j$. Here $\phi_k^{(j)}$ denotes the $j$th derivative of $\phi_k$.

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- Such functions are called completely monotone.

# Proof of chaos under continuous perturbation - 2

- By a classical theorem of Bernstein about completely monotone functions, there is a probability measure $\mu_k$ on $[0, \infty)$ such that

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- Thus, it suffices to prove that $\phi_k(s) \le const. N^{-k}$ for sufficiently large $s$.

# Proof of chaos under continuous perturbation - 3

- Induction from infinity: Since $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$ are independent and uniformly distributed on $\{-1, 1\}^N$ when $t = \infty$, we have $\phi_k(\infty) = const.N^{-k}$.

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- It is possible to manipulate this chain to conclude that

$$\phi_k(s) \leq 2^{-2N} \sum_{\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2} \left(\frac{\boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2}{N}\right)^{2k} \exp\left(2\beta^2 e^{-s}\frac{(\boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2)^2}{N}\right).$$

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- The right hand side is bounded by $const.N^{-k}$ iff $s$ is sufficiently large. (Related to the fact that when $Z \sim N(0,1)$, $\mathbb{E}(e^{\alpha Z^2}) < \infty$ iff $\alpha < 1/2$.) This completes the proof.

# Other results

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- Long list of unsolved questions (actually, almost everything). See in: **Disorder chaos and multiple valleys in spin glasses.** arXiv:0907.3381v1