Yang–Mills for mathematicians

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What is this talk about?

- I am going to give you a heavily compressed exposition of a very long story, and some open problems.
- There will be reading references at the end, if you want to learn more about it.
- Physicists are generally familiar with most of what I'm going to say, but mathematicians are not. This talk is for mathematicians.

Quantum field theories

- Quantum field theories explain interactions between elementary particles and make predictions about their behaviors.
- Encapsulated by the Standard Model.
- Yang–Mills theories are certain kinds of important quantum field theories that constitute the standard model.
- What is a QFT? — This is an open question, not only in mathematics, but also in physics.
- Remarkably, physicists can calculate and make surprisingly accurate predictions using QFTs, without really understanding what these objects are!
- The mathematical construction of quantum field theories — more specifically Yang–Mills theories — is one of the seven millennium problems posed by the Clay Institute.

QFT basics

- Spacetime: $\mathbb{R}^4$. Restricted Lorentz transforms: A group of linear transformations of $\mathbb{R}^4$.
- Poincaré group $\mathcal{P}$ consists of all (a, Λ), where Λ is a restricted Lorentz transformation and $a \in \mathbb{R}^4$.
- $\mathcal{P}$ acts on $\mathbb{R}^4$ as $(a, \Lambda)x = a + \Lambda x$.
- Special relativity: The laws of physics remain invariant under change of coordinates by the action of the Poincaré group.
- A quantum field theory models the behavior of a physical system (e.g. a collection of elementary particles) using:
  - a Hilbert space $\mathcal{H}$, and
  - a (projective) unitary representation U of $\mathcal{P}$ in $\mathcal{H}$.
- Assumptions:
  - To an observer, the state of the system appears as some vector $\psi \in \mathcal{H}$. If ψ is known, we can compute probabilities of events.
  - To a different observer, who is using a coordinate system obtained by the action of (a, Λ) on the coordinate system of the first observer, the state appears as $U(a, \Lambda)\psi$.

Time evolution

- Suppose a stationary observer at spatial location $(0, 0, 0)$ observes the physical system in state ψ at time 0.
- After time t, the system will appear to the observer as being in state $U((-t, 0, 0, 0), \text{Id.})\psi$.
- It can be proved that there is a self-adjoint operator H on $\mathcal{H}$ so that for any t, $U((-t, 0, 0, 0), \text{Id.})\psi = e^{-\mathrm{i}tH}\psi$.
- H is called the Hamiltonian.
- Important to note: $(\mathcal{H}, U)$ describes the behavior of not just one particle, but a system of various kinds of particles, where even the number of particles may not fixed over time. Useful for predicting the outcomes of scattering experiments, for example.

Quantum field

- Suppose we are given $\mathcal{H}$ and U.
- A quantum field φ is a hypothetical function on $\mathbb{R}^4$, which when integrated against a smooth test function on $\mathbb{R}^4$, yields an operator on $\mathcal{H}$.
- To put it more succinctly, it is an operator-valued distribution.
- The quantum field φ related to our physical system is a field that satisfies

$$\varphi(a + \Lambda x) = U(a, \Lambda)\varphi(x)U(a, \Lambda)^{-1}.$$

- The field φ is used for calculating probabilities of events and expected values of various observables. In fact, it becomes the central object of interest in the study of the system.

Wightman axioms

- The most popular approach to giving a fully rigorous definition of a quantum field theory is via the Wightman axioms.
- These axioms are essentially a more precise version of what I described in the previous slides.
- They include some additional conditions (such as 'locality') that must be satisfied by $\mathcal{H}$, U and φ, and some assumptions about the existence and properties of a unique vacuum state $\Omega \in \mathcal{H}$ of our system. (This the lowest eigenstate of H.)
- The axioms give the bare minimum conditions required to avoid physical inconsistencies.
- It has been possible to construct certain simple QFTs, known as free fields, which satisfy the Wightman axioms.
- Free fields describe trivial systems of particles that do not interact with each other.
- *No one has been able to rigorously construct a nontrivial (interacting) QFT in 4D satisfying the Wightman axioms.*

Calculations???

- If we cannot even *define* the theory, how can we calculate?
- Physicists get around this problem by doing perturbative expansions around free fields.
- That is, they assume that the desired QFT is a 'small perturbation' of the free field (which is well-defined), and do a kind of Taylor expansion around it.
- The calculations involve Feynman diagrams and renormalization.
- However, there is a rigorous theorem due to Haag, which says that the Hilbert space for an interacting theory cannot be the same as the Hilbert space for a non-interacting theory.
- So it is not clear how one can justify such a perturbative expansion. *In fact, in most cases it is not clear what the new Hilbert space is!*
- And yet, in many cases, these calculations yield results that match experiments to remarkable degrees of accuracy.

The probabilistic approach (Constructive QFT)

- There is a probabilistic approach to constructing QFTs that satisfy the Wightman axioms. It goes as follows:
  - First, construct a random field ξ on $\mathbb{R}^4$ whose probability law is related to the desired QFT in a certain way. Usually this is a random distribution, and not a random function.
  - ξ is called a Euclidean QFT.
  - Show that ξ satisfies a set of conditions known as the Osterwalder–Schrader axioms.
  - If this is true, then there is a reconstruction theorem that allows us to construct the desired QFT (i.e., $\mathcal{H}$, U, φ and Ω.)
  - In general, the QFT is nontrivial if and only if the field ξ is non-Gaussian.
- The program, initiated in the 60s, was successful in constructing nontrivial QFTs when the dimension of spacetime was reduced from 4 to 2 or 3 — but not yet in dimension 4.
- Notable achievements were the constructions of φ_2^4 and φ_3^4 theories (in spacetime dimensions 2 and 3, respectively).

Yang–Mills theories

- φ^4 theories are mathematically interesting, but describe no real physical system.
- To venture into the real world, one has to consider 4D Yang–Mills theories.
- These are QFTs that describe interactions between real elementary particles.
- The question is completely settled in 2D.
- There was a tremendous amount of work on rigorously constructing Yang–Mills theories in 3D and 4D, by Bałaban and others.
- However, the investigation was inconclusive and the question is still considered to be open.
- Even the first step in the probabilistic approach, namely, the construction of a random field, remains open. We will now talk about that.

Euclidean Yang–Mills theories

- Recall that the first step in the probabilistic approach to constructing QFTs is the construction of a suitable random field, known as a Euclidean QFT.
- For Yang–Mills theories, these random fields are called Euclidean Yang–Mills theories.
- These have not yet been constructed in spacetime dimensions 3 and 4.
- Euclidean Yang–Mills theories are supposed to be scaling limits of lattice gauge theories, which are well-defined discrete probabilistic objects, which I will now discuss.

Lattice gauge theories

- Let d = dimension of spacetime, and G be a matrix Lie group. (Most important: $d = 4$ and $G = SU(2)$ or $SU(3)$.)
- The lattice gauge theory with gauge group G on a finite set $\Lambda \subseteq \mathbb{Z}^d$ is defined as follows.
- Suppose that for any two adjacent vertices $x, y \in \Lambda$, we have a group element $U(x, y) \in G$, with $U(y, x) = U(x, y)^{-1}$.
- Let $G(\Lambda)$ denote the set of all such configurations.
- A square bounded by four edges is called a plaquette. Let $P(\Lambda)$ denote the set of all plaquettes in Λ.
- For a plaquette $p \in P(\Lambda)$ with vertices x_1, x_2, x_3, x_4 in anti-clockwise order, and a configuration $U \in G(\Lambda)$, define

$$U_p := U(x_1, x_2)U(x_2, x_3)U(x_3, x_4)U(x_4, x_1).$$

- The Wilson action of U is defined as

$$S_{\mathrm{W}}(U) := \sum_{p \in P(\Lambda)} \mathrm{Re}(\mathrm{Tr}(I - U_p)).$$

Definition of lattice gauge theory contd.

- Let σ_Λ be the product Haar measure on $G(\Lambda)$.
- Given $\beta > 0$, let $\mu_{\Lambda,\beta}$ be the probability measure on $G(\Lambda)$ defined as

$$d\mu_{\Lambda,\beta}(U) := \frac{1}{Z} e^{-\beta S_{\mathrm{W}}(U)} d\sigma_\Lambda(U),$$

  where Z is the normalizing constant.
- This probability measure is called the lattice gauge theory on Λ for the gauge group G, with inverse coupling strength β.
- An infinite volume limit of the theory is a weak limit of the above probability measures as $\Lambda \uparrow \mathbb{Z}^d$ (may not be unique).

Open problem #1: Yang–Mills existence

- To define the scaling limit of a lattice gauge theory, one has to first define it on the scaled lattice $\epsilon\mathbb{Z}^d$ and then send $\epsilon \to 0$.
- To obtain an interesting limit, one has to vary the parameter β as $\epsilon \to 0$.
- In dimension 3, it is believed β has to scale like a multiple of ϵ^{-1}, and in dimension 4, it is believed that β has to scale like a multiple of $\log(1/\epsilon)$.
- The most interesting gauge groups are non-Abelian Lie groups like $SU(2)$ and $SU(3)$.
- It is not clear what the scaling limit should look like, or what space it should belong to.
- Even if one is able to somehow obtain a scaling limit, it is important to prove that it is nontrivial — meaning that it is a non-Gaussian field (on whatever space it's defined on).
- Finally, one has to construct the actual QFT using this field, via the Osterwalder–Schrader axioms or otherwise.

YM existence: Mathematical literature

- Well-understood in dimension 2. Many contributors.
- In dimensions 3 and 4, long series of papers by Bałaban in the 80s, aiming to prove the existence of subsequential scaling limits. Established results about the behavior of the partition function in the scaling limit.
- However, the problem is still considered to be open in dimensions 3 and 4.
- Recently, probabilists have made exciting new progress in constructing φ_3^4 theory via stochastic quantization (many contributors).

Open problem #2: Mass gap

- Recall the Hamiltonian H of a QFT, and the vacuum state Ω. The vacuum state is the unique (up to scalar multiples) nonzero element of $\mathcal{H}$ that satisfies $H\Omega = 0$.
- The theory is said to have a mass gap if there is some $\mu > 0$ such that any other eigenvalue of H is $\geq \mu$.
- Physically, this means that the particles described by the theory possess nonzero mass.
- If we go through the probabilistic approach, the mass gap question can be shown to be equivalent to the question of exponential decay of correlations in the Euclidean QFT.
- Various Yang–Mills theories — such as 4D Yang–Mills theory with gauge group $SU(3)$ — are supposed to have mass gaps.
- The first step to showing this is to show that the corresponding lattice gauge theories have exponential decay of correlations at large β.

Mass gap: Mathematical literature

- At small β, exponential decay can be proved by well-known techniques from statistical physics (expansions or coupling).
- No general method for large β.

Wilson loops

- Consider a lattice gauge theory on $\mathbb{Z}^d$ with gauge group G.
- Let U be a random configuration of group elements attached to edges, drawn from the probability measure defined by this theory.
- Given a loop γ with directed edges $e_1, \ldots, e_m$, the Wilson loop variable W_γ is defined as

$$W_\gamma := \mathrm{Re}(\mathrm{Tr}(U(e_1)U(e_2)\cdots U(e_m))).$$

- The expected value of W_γ is denoted by $\langle W_\gamma \rangle$.

Quark confinement

- Lattice gauge theories and Wilson loops were introduced by Wilson in 1974 primarily to understand the phenomenon of quark confinement.
- Quarks are elementary particles that bind together to form protons, neutrons, etc.
- Quarks are always bound, and never occur freely in nature. This is known as quark confinement or color confinement.
- Wilson argued that this phenomenon occurs due to a mathematical feature of Yang–Mills theories, that is now called Wilson's area law.

Open problem #3: Quark confinement

- Take any 4D non-Abelian lattice gauge theory.
- Show that for any β, there are constants $C(\beta)$ and $c(\beta)$ such that for any loop γ,

$$|\langle W_\gamma \rangle| \le C(\beta) e^{-c(\beta)\mathrm{area}(\gamma)},$$

  where $\langle W_\gamma \rangle$ is the expected value of the Wilson loop variable W_γ and $\mathrm{area}(\gamma)$ is the minimal surface area enclosed by γ.
- This is known as Wilson's area law, and was argued by Wilson to be the reason behind confinement of quarks.
- Showing for rectangles is good enough.

Quark confinement: Mathematical literature

- There is a general proof at small β by Osterwalder and Seiler (1978).
- Proof at large β for 3D $U(1)$ theory by Göpfert and Mack (1982).
- Disproof at large β for 4D $U(1)$ theory by Guth (1980) and Fröhlich and Spencer (1982). *Therefore in 4D at large β, it is crucial that the gauge group is non-Abelian.*

Gauge-string duality

- In 1997, Maldacena made the remarkable discovery that certain quantum field theories are ‘dual’ to certain string theories.
- Duality means that any calculation in one theory corresponds to some calculation in the other theory.
- Maldacena’s discovery is known as AdS-CFT duality or gauge-string duality or gauge-gravity duality.

Open problem #4: Gauge-string duality in lattice gauge theories

- To establish gauge-string duality for YM theories, one can, for example, try to show that expected values of Wilson loop variables are expressible as integrals over trajectories of strings in a string theory.
- Tremendous activity in physics, but almost nothing on the mathematical side. Possibly because the relevant QFTs are not mathematically well-defined.
- In 2015, I proved such a result for lattice gauge theories at small β — probably the first mathematical theorem in this area. Extended later in a joint work with Jafar Jafarov.
- However, this is a discrete result. It is an open problem to prove such a theorem when β is large. We need to consider large β for passing to the continuum limit.

Strong-weak dualities

- A strong-weak duality is a duality between a physical theory at large β and another physical theory at small β.
- Here, as usual, 'duality' means calculations in one model can be carried out certain 'dual calculations' in the other model. Usually, calculations are easier at small β.
- Earliest example: Kramers–Wannier duality for the 2D Ising model. Many other examples in the literature.
- In a recent preprint, I proved a duality relation for Wilson loop expectations in 4D $\mathbb{Z}_2$ lattice gauge theory which allowed me to calculate Wilson loop expectations to leading order at large β.
- Roughly speaking, the result is that if β is large, and a loop γ has length $\alpha e^{12\beta}$, then $\langle W_\gamma \rangle \approx e^{-2\alpha}$.
- Key step: Express $\langle W_\gamma \rangle$ as an expectation of some other quantity in 4D $\mathbb{Z}_2$ lattice gauge theory at inverse coupling strength $\lambda = -\frac{1}{2} \log \tanh \beta$. Note: $\lambda \to 0$ as $\beta \to \infty$.

Open problem #5: Strong-weak dualities for lattice gauge theories

- Understanding the precise behavior of Wilson loop expectations at large β is crucial for constructing scaling limits.
- The method from my preprint can probably be extended to other Abelian gauge groups. Not clear how to do non-Abelian.
- For non-Abelian theories, there is a conjectured set of such dualities, known as the Montonen–Olive dualities.
- Kapustin and Witten (2007) suggested that the Montonen–Olive dualities are in fact equivalent to the geometric Langlands correspondence.
- Of course, no one knows how to prove anything about any of this.

Where to read about all this

- My preprint "Yang–Mills for probabilists" on arXiv has more details and references for many of the topics presented here. (See also my other preprints on Yang–Mills and lattice gauge theories for references on specific topics.)
- On my website, you will find lecture notes for a course on "Quantum field theory for mathematicians" that I taught recently at Stanford. Introduces the foundations of QFT but not Yang–Mills.
- The above lecture notes are based on a (terrific) forthcoming book by Michel Talagrand that presents QFT for a mathematical audience.
- The developments in stochastic quantization are available in recent papers and preprints of various authors.
- Constructive QFT is explained in the textbook of Glimm and Jaffe, and in many surveys and expositions available online.