Invariant measures and the soliton resolution conjecture

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The focusing nonlinear Schrödinger equation

- A complex-valued function u of two variables x and t, where $x \in \mathbb{R}^d$ is the space variable and $t \in \mathbb{R}$ is the time variable, is said to satisfy a d-dimensional focusing nonlinear Schrödinger equation (NLS) with nonlinearity parameter p if

$$\mathrm{i}\,\partial_t u = -\Delta u - |u|^{p-1}u.$$

- The equation is called "defocusing" if the term $-|u|^{p-1}u$ is replaced by $+|u|^{p-1}u$. If the nonlinear term is absent, we get the ordinary Schrödinger equation.

Mass and energy

- The focusing NLS has two well-known invariants, namely, mass

$$M(u) := \int_{\mathbb{R}^d} |u(x)|^2 dx$$

and energy

$$H(u) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u(x)|^{p+1} dx.$$

- That is, if $u(x,t)$ is a solution of the NLS, then $M(u(\cdot,t))$ and $H(u(\cdot,t))$ remain constant over time.
- List of all conserved quantities unknown (except in $d = 1$, $p = 3$).

Equivalence classes

- We will say that two functions u and v from $\mathbb{R}^d$ into $\mathbb{R}$ are equivalent if
$$v(x) = u(x - x_0)e^{\mathrm{i}\lambda_0}$$
for some $x_0 \in \mathbb{R}^d$ and $\lambda_0 \in \mathbb{R}$.
- Note that if u and v are equivalent in this sense, then $M(u) = M(v)$ and $H(u) = H(v)$.

Solitons

- For the ordinary Schrödinger equation, as well for the defocusing NLS, it is known that in general the solution "radiates to zero" as $t \to \infty$. This means that for every compact set $K \subseteq \mathbb{R}^d$,

$$\lim_{t\to\infty} \int_K |u(x,t)|^2 dx = 0.$$

- In the focusing case this may not happen.
- Demonstrated quite simply by a special class of solutions called "solitons" or "standing waves".
- These are solutions of the form $u(x,t) = v(x)e^{\mathrm{i}\omega t}$, where ω is a positive constant and the function v is a solution of the soliton equation

$$\omega v = \Delta v + |v|^{p-1} v.$$

- Often, the function v is also called a soliton.

Ground state solitons

- When p satisfies the "mass-subcriticality" condition $p < 1 + 4/d$, it is known that there is a unique equivalence class minimizing $H(u)$ under the constraint $M(u) = m$. The minimum energy may be denoted by $E_{\min}(m)$.
- This equivalence class is known as the "ground state soliton" of mass m.
- The ground state soliton has the following description:
  - (Deep classical result) There is a unique positive and radially symmetric solution Q of the soliton equation

    $$\omega Q = \Delta Q + |Q|^{p-1} Q$$

    with $\omega = 1$.
  - For each $\lambda > 0$, let

    $$Q_\lambda(x) := \lambda^{2/(p-1)} Q(\lambda x).$$

    Then each Q_λ is also a soliton (with ω dependent on λ).
  - For each $m > 0$, there is a unique $\lambda(m) > 0$ such that $Q_{\lambda(m)}$ is the ground state soliton of mass m.

The soliton resolution conjecture

- The long-term behavior of solutions of the focusing NLS is still not fully understood.
- One particularly important conjecture, sometimes called the "soliton resolution conjecture", claims that as $t \to \infty$, the solution $u(\cdot, t)$ would look more and more like a soliton, or a union of a finite number of receding solitons.
- The claim may not hold for all initial conditions, but is expected to hold for "generic" initial data.
- Significant progress in recent years (Kenig, Merle, Schlag, Tao, many others ....) but complete solution is still elusive.

Ergodic hypothesis

- Let $\{T_t\}_{t\geq 0}$ be a semigroup of operators on the space of functions from an abstract space $\mathcal{X}$ into $\mathbb{R}$.
- For example, for $f : \mathbb{R}^d \to \mathbb{R}$, $T_t f$ can be the solution of the focusing NLS at time t with initial data f.
- Birkhoff's ergodic theorem: If μ is an ergodic invariant measure for the flow $\{T_t\}_{t\geq 0}$, then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T_s f(x) ds = \int_{\mathcal{X}} f(x) d\mu(x) .$$

- Suggests that one can study the time-averaged long-term behavior of a flow by studying the space-average over an ergodic invariant measure μ.
- Easier to construct invariant measures than proving ergodicity.
- However, any invariant measure decomposes into ergodic components. Therefore Birkhoff's theorem implies that a high probability event for μ occurs within most ergodic components.

Invariant measures for the NLS

- Substantial body of literature on understanding the long-term behavior of the focusing NLS by studying invariants measures.
- Ergodicity has not been proved in any case, as far as I am aware.
- Initiated in a seminal paper of Lebowitz, Rose and Speer (1988).
- Large body of follow-up work in the nineties (Bourgain, McKean, Zhidkov, Vaninsky, Rider, Brydges, Slade, ...).
- Resurgence of interest in recent years (Burq, Tzvetkov, Oh, Staffilani, Bulut,Thomann, Nahmod, ...).
- The focus in the PDE community has mainly been on using invariant measures to prove existence of global solutions.
- As a probabilist, my interest is more in understanding the space-average with respect to these invariant measures, and then appealing to the ergodic hypothesis.

Discrete NLS

- Invariant measures are easier to construct and study in the discrete setting.
- Initial work in:

  Chatterjee, S. and Kirkpatrick, K. (2012). Probabilistic methods for discrete nonlinear Schrödinger equations. *Comm. Pure Appl. Math.*, **65** no. 5, 727–757.
- This talk is based on:

  Chatterjee, S. (2014). Invariant measures and the soliton resolution conjecture. *Comm. Pure Appl. Math.*, **67** no. 11, 1737–1842.

How to discretize?

- Let $\mathbb{T}_n^d = \{0, 1, \ldots, n-1\}^d = (\mathbb{Z}/n\mathbb{Z})^d$ be the discrete torus of width n.
- Imagine this set embedded in $\mathbb{R}^d$ as $h\mathbb{T}_n^d$, where $h > 0$ is the mesh size.
- $h\mathbb{T}_n^d$ is a discrete approximation of the box $[0, nh]^d$.
- Define the discrete Laplacian on $h\mathbb{T}_n^d$:

$$\Delta u(x) = \frac{1}{h^2} \sum_{y \text{ is a nhbr of } x} (u(y) - u(x)) \,.$$

- Focusing DNLS on $h\mathbb{T}_n^d$:

$$\mathrm{i}\, \frac{du}{dt} = -\Delta u - |u|^{p-1} u \,.$$

Mass and energy

- Luckily, the DNLS is also a Hamiltonian flow.
- The discrete mass and energy of a function $u : h\mathbb{T}_n^d \to \mathbb{C}$, defined below, are conserved quantities for this flow:

$$M(u) := h^d \sum_x |u(x)|^2,$$

and

$$H(u) := \frac{h^d}{2} \sum_{x,\, y \text{ nhbrs}} \left| \frac{u(x) - u(y)}{h} \right|^2 - \frac{h^d}{p+1} \sum_x |u(x)|^{p+1}.$$

Microcanonical ensemble

- Fixing $\epsilon > 0$, $E \in \mathbb{R}$ and $m > 0$, define

$$S_{\epsilon,h,n}(E, m) := \{u : |M(u) - m| \leq \epsilon, |H(u) - E| \leq \epsilon\}.$$

- In words, $S_{\epsilon,h,n}(E, m)$ is the set of all functions on $h\mathbb{T}_n^d$ with mass $\approx m$ and energy $\approx E$.
- By Liouville's theorem, the uniform probability measure on $S_{\epsilon,h,n}(E, m)$ is an invariant measure for the DNLS flow on $h\mathbb{T}_n^d$.
- Let f be a random function drawn from this uniform probability measure.
- A high probability event for f reflects the long-term behavior of the DNLS flow in "most" ergodic components of this invariant measure.
- Main question: What is the behavior of f? Does it look like a soliton in some limit?

Proof of a statistical version of the soliton resolution conjecture for the discrete NLS

Theorem (C., 2014)

Suppose that $1 < p < 1 + 4/d$. *Fix* E *and* m *such that* $E > E_{\min}(m)$. *Let* Q *be the ground state soliton of mass* m. *Let* f *be a uniform random choice from the set* $S_{\epsilon,h,n}(E,m)$. *Then for any* $\delta > 0$,

$$\lim_{h\to 0} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\left(\inf_{\substack{x_0\in\mathbb{R}^d \\ \lambda_0\in\mathbb{R}}} \max_{x\in h\mathbb{T}_n^d} |f(x) - Q(x-x_0)e^{\mathrm{i}\lambda_0}| > \delta \right) = 0\,.$$

Important note: It is guaranteed by construction that $M(f) \approx m$ and $H(f) \approx E > E_{\min}(m)$. So f cannot be close to the ground state soliton in the H^1 norm. The theorem says that in an appropriate limit, f is close to the ground state soliton in the L^∞ norm.

Proof sketch

- First, prove that for h fixed, f is close to a discrete ground state soliton with high probability. This is the probabilistic part of the proof. Will say more about this in the next few slides.
- The second step is to prove that discrete ground state solitons converge to the continuum ground state soliton as the mesh size tends to zero. This is the analytic part of the proof, involving delicate estimates about discrete Green's functions and discrete versions of various classical inequalities (Littlewood-Paley decompositions, Hardy-Littlewood-Sobolev inequality of fractional integration, Gagliardo-Nirenberg inequality, etc.) with constants that do not blow up as the mesh size goes to zero. Also need discrete concentration compactness to prove stability of discrete solitons (which is trickier than concentration compactness in the continuum), and exponential decay of discrete solitons.

Convergence to discrete solitons

- Let $E_{\min}(m, h)$ denote the minimum possible energy of a function of mass m, in the discrete setting with mesh size h and $n \to \infty$.
- The proof shows that there exists $m^* < m$ such that with high probability, the random function f is close to a discrete ground state soliton of mass m^*.
- Later, it is shown that $m^* \to m$ as $h \to 0$.

Convergence to discrete solitons (contd.)

- Fix $\delta > 0$ and let $U := \{x \in h\mathbb{T}_n^d : |f(x)| > \delta\}$.
- Let $f^v(x) := f(x)$ if $x \in U$, 0 otherwise.
- Let $f^i(x) := f(x) - f^v(x)$.
- The superscripts v and i stand for "visible" and "invisible": f^v is the visible part of f and f^i is the invisible part of f.
- Suffices to show that with high probability, f^v is close to a ground state soliton in the large n limit.
- By the stability of discrete solitons, suffices to prove that with high probability, $M(f^v) \approx m^*$ and $H(f^v) \approx E^*$ for some $m^* \in [0, m]$ and $E^* = E_{\min}(m^*, h)$.

Convergence to discrete solitons (contd.)

- Recall that f is drawn uniformly at random from the set of all u with $M(u) \approx m$ and $H(u) \approx E$.
- Therefore, for any m', E',

$$\mathbb{P}(M(f^v) \approx m',\, H(f^v) \approx E')$$
$$= \frac{\mathrm{Vol}(\{u : M(u^v) \approx m',\, H(u^v) \approx E',\, M(u) \approx m,\, H(u) \approx E\})}{\mathrm{Vol}(\{u : M(u) \approx m,\, H(u) \approx E\})}$$

- Let $V(m', E')$ denote the numerator and V denote the denominator.
- Need to show that there exists $m^* \in [0, m]$ and $E^* = E_{\min}(m^*, h)$ such that

$$\sum_{(m',E') \not\approx (m^*,E^*)} V(m', E') \ll V\,.$$

- Need upper bound on $V(m', E')$ and lower bound on V (that should actually closely match the true values).

Convergence to discrete solitons (contd.)

- The lower bound on V is obtained by guessing m^* and E^* and then using $V \geq V(m^*, E^*) \geq$ the volume of a neighborhood of a discrete soliton of mass m^* and energy E^*.
- Let us now see how to get an upper bound on $V(m', E')$. Assume $h = 1$ for simplicity.
- Suppose that $M(u^v) \approx m'$ and $H(u^v) \approx E'$. Then $M(u^i) \approx m - m'$, and $H(u^i) \approx E - E'$.
- Now, $|u^i(x)| \leq \delta$ everywhere. So

$$\sum |u^i(x)|^{p+1} \leq \delta^{p-1} \sum |u^i(x)|^2 = \delta^{p-1} M(u^i) \approx \delta^{p-1} m\,.$$

- If δ is small, this implies that

$$H(u^i) \approx \frac{1}{2} \sum_{x,\, y \text{ nhbrs}} |u^i(x) - u^i(y)|^2\,.$$

- Lastly, observe that

$$|U| = |\{x : |f(x)| > \delta\}| \leq \delta^{-2} \sum |u(x)|^2 \approx \delta^{-2} m\,.$$

Convergence to discrete solitons (contd.)

- Thus,

$$V(m', E') \le \mathrm{Vol}(\{u : \exists U,\ |U| \le \delta^{-2} m,\ \sum_{x \notin U} |u(x)|^2 \approx m - m',$$
$$\frac{1}{2} \sum_{\substack{x,\, y \text{ nhbrs} \\ x, y \notin U}} |u(x) - u(y)|^2 \approx E - E'\})$$

$$\le \sum_{U : |U| \le \delta^{-2} m} \mathrm{Vol}(\{u : \sum_{x \notin U} |u(x)|^2 \approx m - m',$$
$$\frac{1}{2} \sum_{\substack{x,\, y \text{ nhbrs} \\ x, y \notin U}} |u(x) - u(y)|^2 \approx E - E'\}) .$$

Convergence to discrete solitons (contd.)

The last displayed item in the previous slide is estimated using the following large deviation principle.

Theorem (C., 2014)

Let ξ be a random function chosen uniformly from the set of all functions $u : \mathbb{T}_n^d \to \mathbb{C}$ that satisfy $\sum |u(x)|^2 = 1$. Then for any $\alpha \in (0, 2d)$,

$$\lim_{n\to\infty} \frac{1}{n^d} \log \mathbb{P}\biggl(\sum_{x,\, y \ \textit{nhbrs}} |\xi(x) - \xi(y)|^2 \le \alpha \biggr) = -\Psi_d(\alpha)\,,$$

where

$$\Psi_d(\alpha) = \sup_{0<\gamma<1} \int_{[0,1]^d} \log\biggl(1 - \gamma + \frac{4\gamma}{\alpha} \sum_{i=1}^d \sin^2(\pi x_i)\biggr) dx_1 \cdots dx_d\,.$$

When $\alpha > 2d$, the same result holds after replacing $\le$ by $\ge$.

Convergence to discrete solitons (wrapping up)

- The large deviation principle does not follow from standard techniques, because of localization phenomena.
- The proof involves Fourier analysis on the discrete torus, since $\sum |\xi(y) - \xi(y)|^2$ can be elegantly written as a linear combination of Fourier coefficients.
- In certain regimes, a small number low Fourier coefficients become exceedingly large (localization).
- The fact that $V(m', E')$ is maximized at some (m^*, E^*), where $E^* = E_{\min}(m^*, h)$, follows from analyzing the large deviation rate function displayed in the previous slide. This is, of course, the central reason why f is close to a soliton. Beyond this calculation involving the rate function, I don't have an intuition for why this happens.

That is all. Thanks for your attention.