An invariance principle for 1D KPZ

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(Joint work with Arka Adhikari)

The 1D KPZ equation

- The Kardar–Parisi–Zhang (KPZ) equation models the growth of a generic randomly growing surface.
- In this talk, we will restrict ourselves to one-dimensional surfaces.
- If $f(x,t)$ is the height of a 1D surface at time $t \in \mathbb{R}_+$ and location $x \in \mathbb{R}$, we says that it grows according to the (standardized) KPZ equation if

$$\partial_t f = \partial_x^2 f + (\partial_x f)^2 + \xi,$$

where ξ is a random field known as space-time white noise.

White noise

- The white noise field ξ is a random distribution on $\mathbb{R}_+ \times \mathbb{R}$, with the property that for any Borel set $A \subseteq \mathbb{R}_+ \times \mathbb{R}$ with finite Lebesgue measure,
$$\xi(A) := \int_A \xi(x,t)\, dx\, dt \sim \mathcal{N}(0, \mathrm{area}(A)).$$
- A consequence of the above is that if A and B are disjoint, then $\xi(A)$ and $\xi(B)$ are independent.
- ξ can be rigorously defined. The main problem in rigorously solving the KPZ equation comes from trying to define $(\partial_x f)^2$.
- We will talk about how to make sense of the KPZ equation. But first, let us briefly discuss the physical intuition behind the KPZ equation.

The Edwards–Wilkinson model

- The Edwards–Wilkinson (EW) surface growth model, introduced by Edwards and Wilkinson in 1981, goes as follows.
- Space is now discretized, replacing $\mathbb{R}$ by $\mathbb{Z}$. Time remains continuous.
- At each $x \in \mathbb{Z}$, there is a Poisson clock that rings at rate 1.
- Whenever the clock rings, the height at x is instantaneously updated to the average of the heights at $x-1$, x, and $x+1$, plus a random noise.

Stochastic heat equation (SHE) with additive noise

- Upon suitably scaling space and time, and sending the variance of the noise to zero, the Edward–Wilkinson surface converges to a scaling limit known as the Stochastic Heat Equation (SHE) with additive noise:

$$\partial_t f = \partial_x^2 f + \xi.$$

- Note that this is the KPZ equation without the problematic $(\partial_x f)^2$ term.

Physical justification

- Although natural and appealing, the EW model was found to be inadequate for modeling the behaviors of growing surfaces.
- This was the motivation for Kardar, Parisi and Zhang to introduce their model in 1986.
- The KPZ logic was that the main inadequacy of the EW model lies in its feature that the growth at a point is a linear function of the gradient of the surface at that point (plus random noise).
- Instead, they proposed that one should consider nonlinear effects.
- Since $(\partial_x f)^2$ is the simplest nonlinear effect, they augmented the SHE with this term. This is the rough justification for the KPZ equation.

Solutions of the 1D KPZ equation

- We now know how to construct solutions of the 1D KPZ equation using a variety of techniques, such as:
  - The Cole–Hopf solution (Bertini, Giacomin, ...).
  - Regularity structures (Hairer).
  - Paracontrolled distributions (Gubinelli, Perkowski, ...).
  - Energy solutions (Gonçalves, Jara, ...).
  - Renormalization group (Kupiainen, Marcozzi, ...).
- Of these, the Cole–Hopf solution is the earliest, and the one most relevant for us.

The Cole–Hopf solution

- The 1D stochastic heat equation with multiplicative noise is the solution of the differential equation

$$\partial_t Z = \partial_x^2 Z + Z\xi.$$

- Note that this is just like the SHE with additive noise, except that the white noise ξ is now multiplied by Z.
- It turns out that with suitable initial data, the solution can be rigorously defined and is positive everywhere.
- Although the solution is not smooth, $\partial_x^2 Z$ makes sense as a random distribution.
- Let $f := \log Z$. A formal calculation using the rules of stochastic calculus shows that f must be a solution of the KPZ equation.
- Although this formal calculation is not rigorous — because $(\partial_x f)^2$ has no rigorous meaning — this is generally accepted as a solution of 1D KPZ. It is known as the Cole–Hopf solution.

Weak universality conjecture

- The weak universality conjecture for the 1D KPZ equation says that any 1D interface growth process that is driven by microscopic fluctuations, where the heights at neighboring points have a nontrivial effect on the growth of the height at a point, should converge to a solution of the KPZ equation in some suitable scaling limit.
- This is admittedly rather ill-posed, but it is one of the things that make the KPZ equation an object of central interest.

Evidence in favor

- There is now considerable evidence in favor of the weak universality conjecture, mainly in the form of rigorously proved convergences of various discrete growth models to the KPZ equation in a suitable space-time scaling limit. Some examples are:
  - Asymmetric exclusion processes in the weakly asymmetric limit (Bertini–Giacomin, Amir–Corwin–Quastel, Dembo–Tsai, Yang).
  - q-TASEP processes (Borodin–Corwin, Corwin–Tsai).
  - Directed random polymers in the intermediate disorder regime (Alberts–Khanin–Quastel, Moreno Flores–Quastel–Remenik).
  - A large class of stationary, weakly asymmetric, conservative particle systems (Goncalves–Jara). The limit here is the energy solution of the KPZ equation, which was later shown to be unique by Gubinelli–Perkowski. A more general result along the same line was proved later by Diehl–Gubinelli–Perkowski.
  - KPZ equation with smoothed nonlinearity, taken to the limit where smoothing is removed (Funaki–Quastel).
  - The KPZ equation with $(\partial_x f)^2$ replaced by $F(\partial_x f)$ for some general nonlinear function F, under appropriate limits of scaling space and time (Hairer–Quastel, Hairer–Xu).

An attempt at a general formulation of weak universality

- Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a function, which we will call the 'driving function' for our surface growth process.
- Let $z(x,t)$, $t \in \mathbb{Z}_+$, $x \in \mathbb{Z}$ be a collection of i.i.d. random variables with finite moment generating function m.
- Given n, consider a growing random surface f_n defined via the recursion

$$f_n(x, t+1) = \psi(f_n(x-1,t), f_n(x+1,t)) + n^{-1/4} z(x, t+1),$$

  with $f_n(x,0) = 0$ for all x.
- Does f exhibit KPZ behavior when n is large, after rescaling space and time?
- The answer is yes, upon assuming certain properties of ψ.

Assumptions on the driving function

- We assume:
  - Equivariance under constant shifts. For all $x, y, c \in \mathbb{R}$, $\psi(x + c, y + c) = \psi(x, y) + c$.
  - Regularity. ψ is smooth in a neighborhood of the origin.
  - Symmetry. $\psi(x, y) = \psi(y, x)$ for all x, y.
- The above assumptions are natural from a physical point of view.
- An example of a ψ satisfying the above assumptions is

$$\psi(u, v) = \frac{u + v}{2} + (u - v)^2,$$

  which represents a 'discrete version' of KPZ growth.
- Another example, considered in the original paper of Kardar, Parisi and Zhang, is

$$\psi(u, v) = \frac{u + v}{2} + \sqrt{1 + (u - v)^2}.$$

Two constants

- We need to define two constants. The first constant is

$$\beta := \partial_1^2 \psi(0,0),$$

where ∂_1^2 denotes the second partial derivative in the first coordinate.

- For $x \in \mathbb{Z}$ and $t \in \mathbb{Z}_+$, let $p(x,t)$ denote the probability that a simple symmetric random walk on $\mathbb{Z}$, started at 0 at time 0, is at x at time t. Let

$$\Delta(x,t) := p(x+1,t) - p(x-1,t).$$

- Next, define

$$c := \frac{1}{24}\partial_1^4 \psi(0,0) + \frac{\beta^3}{12}.$$

- Finally, we define the second constant

$$V := c\left[\sum_{x\in\mathbb{Z}}\sum_{t=0}^{\infty} \Delta(x,t)^4(\mu_4 - \mu_2^2) + \left(\sum_{x\in\mathbb{Z}}\sum_{t=0}^{\infty} \Delta(x,t)^2 \mu_2\right)^2\right],$$

where μ_k denotes the k^{th} moment of the noise variables.

Renormalized surface

- Having defined β and V, we now define the rescaled and renormalized surface growth process $\tilde{f}_n : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$.
- For any $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ such that x is an integer multiple of $n^{-1/2}$ and t is an integer multiple of n^{-1}, let

$$\begin{aligned}\tilde{f}_n(x,t) &:= f_n(\sqrt{n}x, nt) \\ &\qquad - \biggl(V + \frac{1}{2}\beta n^{1/2}\mu_2 + \frac{1}{6}\beta^2 n^{1/4}\mu_3 \\ &\qquad\qquad + \frac{1}{24}\beta^3(\mu_4 - 3\mu_2^2) + n\psi(0,0)\biggr)t.\end{aligned}$$

- Note that the renormalization term depends only on the first four moments of the noise variables.
- For all other (x, t), define $\tilde{f}_n(x, t)$ by linear interpolation.

Main result

Theorem (Adhikari and C., 2022)

If $\beta \neq 0$, then the $C(\mathbb{R} \times \mathbb{R}_+)$-valued random function $\exp(\beta \tilde{f}_n)$ *converges in law as $n \to \infty$ to a solution Z of the stochastic heat equation with multiplicative noise*

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \sqrt{2\mu_2}\beta Z \xi, \quad Z(\cdot, 0) \equiv 1,$$

where ξ is standard space-time white noise. This means that $\tilde{f}_n$ converges in law to the Cole–Hopf solution of the KPZ equation. The topology on $C(\mathbb{R} \times \mathbb{R}_+)$ that we use here is the topology of uniform convergence on compact sets. If $\beta = 0$, then $\tilde{f}_n$ converges in law to a solution f of the stochastic heat equation with additive noise

$$\partial_t f = \frac{1}{2}\partial_x^2 f + \sqrt{2\mu_2}\xi, \quad f(\cdot, 0) \equiv 0.$$

Directed polymers

- The proof of the theorem proceeds by comparing our general growth process with the special case of directed polymer growth.
- Let $z(x,t)$ be i.i.d. $N(0,1)$ random variables, $t \in \mathbb{Z}_+$, $x \in \mathbb{Z}$.
- Fix a parameter $\theta \in \mathbb{R}$, called the 'inverse temperature'.
- Define

$$f_\theta(x,t) := \log\left[\frac{1}{2^{t-1}} \sum_{P \in \mathcal{P}_t} \exp\left(\theta \sum_{i=0}^{t-1} z_{x+p_i, t-i} - \frac{\theta^2 t}{2}\right)\right],$$

  where $\mathcal{P}_t$ is the set of all $P = (p_0, \ldots, p_{t-1}) \in \mathbb{Z}^t$ such that $p_0 = 0$ and $|p_i - p_{i-1}| = 1$ for each i, where $|\cdot|$ is the Euclidean norm. (That is, $\mathcal{P}_t$ is the set of all simple random walk paths of length $t-1$ starting from 0.)
- $f_\theta(x,t)$ denotes the height of the surface at time t and location x in the directed polymer model at inverse temperature θ, if we start from the all zero initial condition.

Scaling limit of directed polymers

- Suppose now that we rescale space and time, and simultaneously rescale θ as follows: Take some $\lambda \in \mathbb{R}$ and $n \geq 1$. Define $f_n^{\text{poly}} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ as

$$f_n^{\text{poly}}(t,x) := \frac{1}{\lambda} f_{\lambda n^{-1/4}}([nt], [\sqrt{n}x]).$$

- Alberts, Khanin and Quastel proved that h converges weakly (in the sup-norm topology) as $n \to \infty$ to the Cole–Hopf solution of the rescaled KPZ equation

$$\partial_t f = \partial_x^2 f + \frac{\lambda}{2}(\partial_x f)^2 + \xi.$$

- The particular way to scale the inverse temperature with space and time is known as the intermediate disorder regime.

Plan of attack

- We decompose our surface growth process f_n as

$$f_n = f_n^{\text{poly}} + Y_n + \delta_n,$$

  where
  - f_n^{poly} is the log-partition function for the directed polymer model at inverse temperature $\beta n^{-1/4}$,
  - Y_n is a nonlinear function of the polymer process that acts as a renormalization term, and
  - δ_n is an error term which is $o(1)$ as $n \to \infty$.
- The main part of the argument rests on carefully choosing the renormalization term Y_n, and then showing that $\delta_n \to 0$ and $Y_n(x, t)$ behaves like a constant multiple of t.
- These are proved by induction on t, fixing n.

Some details

- Recall: $p(x,t)$ is the probability that a simple symmetric random walk on $\mathbb{Z}$, started at 0 at time 0, is at x at time t, and

$$\Delta(x,t) := p(x+1,t) - p(x-1,t).$$

- Fixing some $\epsilon \in (0, 1/100)$, define

$$K_n(x,t) := \frac{1}{2}\sum_{z\in\mathbb{Z}}\sum_{t-n^\epsilon\le s\le t} \Delta(x-z,t-s)\xi(z,s).$$

- We define

$$Y_n(x,t) := \frac{16c}{\beta^4}\sum_{z\in\mathbb{Z}}\sum_{s=1}^{t} p(x-z,t-s)K_n(z,s)^4,$$

where c and β are constants defined earlier.

Details, contd.

- Having defined Y_n, we define

$$\delta_n(x,t) := f_n(x,t) - f_n^{\mathrm{poly}}(x,t) - Y_n(x,t).$$

- Fixing any $a, b > 0$, we then show that on an event E_n of probability $\to 1$ as $n \to \infty$,

$$|\delta_n(x,t)| \le n^{-1-\epsilon/2} t$$

  for all $(x,t) \in [-an, an] \times [0, bn]$.
- Having suitably defined E_n, we prove the above claim by induction on t. (Observe that $\delta(x,0) = 0$ for all x, which allows us to start the induction.)

Details, contd.

- The induction step uses only properties of the f_n^{poly} process, which makes it tractable.
- The proof is then completed by showing that $Y_n(x,t) \approx \mathbb{E}(Y_n(x,t))$ with high probability, and then showing that $\mathbb{E}(Y_n(x,t)) \approx$ a constant multiple of t.