

Some progress on 2D KPZ

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(joint work with Alex Dunlap)

The $(d + 1)$ -dimensional Kardar–Parisi–Zhang (KPZ) equation

- ▶ “Universal” model of a growing random surface $h(t, x)$ ($t \geq 0, x \in \mathbb{R}^d$), satisfying

$$\partial_t h = \nu \Delta h + \frac{\lambda}{2} |\nabla h|^2 + \sqrt{D} \dot{W},$$

where $\nu, \lambda, D > 0$ and \dot{W} is standard space-time white noise.

- ▶ Formally, \dot{W} is a centered Gaussian field satisfying

$$\mathbb{E}(\dot{W}(t, x) \dot{W}(t', x')) = \delta(t - t') \delta(x - x').$$

- ▶ Introduced by Kardar, Parisi and Zhang in 1986. Huge literature.
- ▶ Difficulty: ∇h is expected to be a random distribution, and therefore $|\nabla h|^2$ is undefined.

The case $d = 1$

- ▶ Most of the literature is about $d = 1$.
- ▶ A large number of discrete models have been shown to converge to the $(1 + 1)$ -dimensional KPZ equation in the scaling limit, justifying its claim to universality.
- ▶ The **Cole–Hopf solution** is a formal solution of the KPZ equation, obtained as $h = (2\nu/\lambda) \log u$, where u solves the stochastic heat equation (SHE) with multiplicative noise:

$$\partial_t u = \nu \Delta u + (\lambda \sqrt{D}/2\nu) u \dot{W}.$$

- ▶ A **direct approach** to solving the KPZ equation in $d = 1$ was provided by Hairer's theory of **regularity structures**.
- ▶ Other direct approaches: **paracontrolled distributions** by Gubinelli, Imkeller and Perkowski, **energy solutions** by Gonçalves and Jara, **renormalization approach** by Kupiainen.

Problem with the Cole–Hopf solution in $d \geq 2$

- ▶ The solutions to the multiplicative SHE are now well-understood in all dimensions.
- ▶ The case $d \geq 3$ was analyzed by Mukherjee, Shamov and Zeitouni (2016), and the case $d = 2$ by Caravenna, Sun and Zygouras (2017), and also by Feng (2016), following early contributions of Bertini and Cancrini (1998).
- ▶ If u is a solution to the multiplicative SHE, then $h = (2\nu/\lambda) \log u$ is a formal solution of the KPZ equation in any dimension.
- ▶ However, when $d \geq 2$, u is a distribution rather than a function. So it is not clear how to define $\log u$.

Difficulty in $d = 2$

- ▶ The $d = 1$ case has a special advantage: In the language of PDEs, it is **subcritical**, and in the language of renormalization, it is **ultraviolet superrenormalizable**.
- ▶ All methods for solving the KPZ equation in $d = 1$ depend crucially on the above feature.
- ▶ What this means is that there is a way to rescale space and time such that the coefficient of $|\nabla h|^2$ can be made as small as we like while keeping the other two coefficients fixed.
- ▶ This is also possible in $d \geq 3$, with a different rescaling. The system is called **infrared superrenormalizable**. Recently analyzed by Magnen and Unterberger (2018).
- ▶ The case $d = 2$ is **critical**, in PDE language. In the language of renormalization, it is **not superrenormalizable**.
- ▶ To summarize, **neither the Cole–Hopf solution nor the direct approaches such as regularity structures** are expected to work in $d = 2$. In this sense, $d = 2$ is the hardest dimension.

Mollified equation

- ▶ Approaches to solving the KPZ equation usually begin in the following way.
- ▶ Let \dot{W}_ε be the white noise \dot{W} mollified by convolving with a spatial mollifier ρ^ε , where $\rho^\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$, and ρ is a compactly supported smooth function.
- ▶ Consider the mollified equation

$$\partial_t h_\varepsilon = \nu \Delta h_\varepsilon + \frac{\lambda}{2} |\nabla h_\varepsilon|^2 + \sqrt{D} \dot{W}_\varepsilon.$$

- ▶ Usually, this equation is solvable in a traditional sense.
- ▶ The question is, what happens as $\varepsilon \rightarrow 0$?

- ▶ **The renormalization viewpoint:** The parameters ν , λ and D need not be fixed parameters; they are free to depend on ε . The main quest is to vary them in such a way so as to get a meaningful/interesting limit object as $\varepsilon \rightarrow 0$.
- ▶ Additionally, for the KPZ equation, one may need to subtract off a constant C_ε depending on ε .
- ▶ In the context of making numerical predictions, “meaningful/interesting” means “yields numbers that match experiment”.

Renormalized scaling limit

- ▶ To summarize, we need to consider the equation

$$\partial_t h_\varepsilon = \nu_\varepsilon \Delta h_\varepsilon + \frac{\lambda_\varepsilon}{2} |\nabla h_\varepsilon|^2 + \sqrt{D_\varepsilon} \dot{W}_\varepsilon - C_\varepsilon,$$

where ν_ε , λ_ε , D_ε and C_ε are constants depending on ε defined in such a way that the solution h_ε converges to some interesting limit as $\varepsilon \rightarrow 0$.

- ▶ Note that there may be multiple ways to vary ν_ε , λ_ε , D_ε and C_ε with ε that give interesting limits.
- ▶ When $d = 1$, it turns out that an interesting limit can be obtained if ν , λ and D are kept fixed and C_ε is made to blow up appropriately as $\varepsilon \rightarrow 0$ (so that h_ε always has mean zero).
- ▶ In fact this is the Cole–Hopf solution in $d = 1$.
- ▶ In $d \geq 3$, Magnen and Unterberger (2018) scaled λ like $\varepsilon^{(d-2)/2}$, blew up C_ε appropriately, and kept ν and D fixed to get a Gaussian scaling limit.

Theorem (C. and Dunlap (2018))

Consider the mollified KPZ equation on a 2D torus. Suppose that we keep ν and D fixed, blow up C_ε appropriately, and set $\lambda_\varepsilon = \lambda |\log \varepsilon|^{-1/2}$ for some $\lambda > 0$. That is, consider

$$\partial_t h_\varepsilon = \nu \Delta h_\varepsilon + \frac{\lambda}{2\sqrt{|\log(1/\varepsilon)|}} |\nabla h_\varepsilon|^2 + \sqrt{D} \dot{W}_\varepsilon - C_\varepsilon.$$

If λ is small enough, the solution h_ε converges in law to a limiting random distribution along some sequence of $\varepsilon \rightarrow 0$. *Moreover, this limit is not the same as the one obtained by simply putting $\lambda = 0$.*

The Family–Vicsek scaling

- ▶ The above theorem is the first rigorous result about a scaling limit for 2D KPZ.
- ▶ However, our renormalization of parameters is probably not the only possible renormalization that gives an interesting limit.
- ▶ Numerical simulations suggest that it may also be possible to obtain a **function-valued scaling limit** by taking $\nu \sim \varepsilon^{2-z}$, $\lambda \sim \varepsilon^{2-z-\alpha}$, and $D \sim \varepsilon^{2+2\alpha-z}$ for certain exponents α and z . This is known as the **Family–Vicsek scaling**.
- ▶ Scaling arguments suggest that $\alpha + z = 2$. If we assume this, then we obtain the scaling $\nu \sim \varepsilon^\alpha$, $\lambda \sim 1$, and $D \sim \varepsilon^{3\alpha}$.
- ▶ This amounts to considering the KPZ equation with *fixed* values of the parameters, and considering the solution multiplied by ε^α on a short time scale $t \sim \varepsilon^\alpha$.
- ▶ There is no consensus about the value of α . Mathematically, the problem is **wide open**.

- ▶ Our theorem is essentially a tightness result.
- ▶ We prove tightness of $\{h_\varepsilon\}_{\varepsilon>0}$ on an appropriate space of distributions.
- ▶ This is a kind of Besov space, used previously by Hairer and others in the study of KPZ and other systems.
- ▶ A convenient criterion for tightness on these spaces was recently provided by Furlan and Mourrat (2017), which we use.

- ▶ For simplicity, let me just sketch the proof of tightness for the family of random variables

$$X_\varepsilon := \int h_\varepsilon \phi,$$

where ϕ is some compactly supported smooth function on the torus.

- ▶ Recall that we have adjusted the renormalization constant C_ε suitably, so that $\mathbb{E}(h_\varepsilon(t, x)) = 0$ for all t, x . Thus, $\mathbb{E}(X_\varepsilon) = 0$.
- ▶ So it suffices to prove that $\sup_{0 < \varepsilon < 1} \mathbb{E}(X_\varepsilon^2) < \infty$.

Proving tightness

- ▶ The first step is to write h_ε as $\log u_\varepsilon$, where u_ε is the solution of the mollified multiplicative SHE.
- ▶ Thus,

$$X_\varepsilon = \int \phi \log u_\varepsilon.$$

- ▶ u_ε has a representation in terms of a [Feynman–Kac formula](#).
- ▶ Due to this explicit representation, we can use [Malliavin calculus](#) to calculate

$$\frac{\partial}{\partial \lambda} \mathbb{E}(X_\varepsilon^2).$$

- ▶ If we can show that the above derivative is uniformly bounded when $0 < \varepsilon < 1$ and $0 \leq \lambda \leq \lambda_0$ for some $\lambda_0 > 0$, and also show that $\sup_{0 < \varepsilon < 1} \mathbb{E}(X_\varepsilon^2) < \infty$ when $\lambda = 0$, then we will reach our objective of showing that $\sup_{0 < \varepsilon < 1} \mathbb{E}(X_\varepsilon^2) < \infty$ when $0 \leq \lambda \leq \lambda_0$.

Recursion

- ▶ Unfortunately, the computation of $\partial\mathbb{E}(X_\varepsilon^2)/\partial\lambda$ yields an expectation of the intersection time of two random paths chosen according to a **continuum polymer measure**, which is Wiener measure with a random tilt.
- ▶ We do not know how to calculate expectations of such intersection times.
- ▶ But when $\lambda = 0$, the polymer measure reduces to just 2D Wiener measure. We know how to calculate expectations of Brownian intersection times.
- ▶ Thus, we can again take derivative with respect to λ , and hope to get a bound for $\partial^2\mathbb{E}(X_\varepsilon^2)/\partial\lambda^2$.
- ▶ But this yields an expectation of a polynomial in the pairwise intersection times of **four** random paths from a polymer measure.
- ▶ One can keep doing this, but it just gets more and more complicated. In k steps, complexity grows like e^{Ck^2} .

Solving the recursion

- ▶ To solve the recursion problem, the first step is to reparametrize λ as $\sqrt{\beta}$, and take derivatives with respect to β . Fixing ε , let $g(\beta) := \mathbb{E}(X_\varepsilon^2)$.
- ▶ We want to control $g(\beta)$ by controlling $g'(\beta)$, and control $g'(\beta)$ by controlling $g''(\beta)$, and so on.
- ▶ The problem is that these derivatives become hopelessly complicated.
- ▶ So we find a way around this problem:
 - ▶ Set $g_0 = g$. Instead of bounding g' , find a nicer function g_1 such that $|g'(\beta)| \leq g_1(\beta)$.
 - ▶ To bound g_1 , we seek to bound g_1' . So we find a nicer function g_2 such that $|g_1'(\beta)| \leq g_2(\beta)$.
 - ▶ We keep going like this, and end up with a sequence of functions g_1, g_2, \dots such that $|g_k'(\beta)| \leq g_{k+1}(\beta)$ for each k .
 - ▶ The nice feature of this sequence is that the complexity grows like e^{Ck} .

Completing the proof

- ▶ For each k , $g_k(0)$ can be obtained by a Brownian computation.
- ▶ Finally,

$$|g(\beta)| \leq \sum_{k=0}^{\infty} \frac{\beta^k}{k!} |g_k(0)|.$$

- ▶ Since the complexity of g_k increases only exponentially, and the k th moment of intersection times grow like $k!$, we can bound $|g_k(0)| \leq C^k k!$ for some C .
- ▶ Thus, for β sufficiently small, we get a finite bound.
- ▶ Moreover, it turns out that the bound does not blow up as $\varepsilon \rightarrow 0$.
- ▶ A similar technique was used for the proof of the disorder chaos conjecture for the Sherrington–Kirkpatrick model of spin glasses (C., 2009).

Contribution of the nonlinearity

- ▶ The last sentence in our theorem asserts that the scaling limit we obtain is not the same as the one obtained by putting $\lambda = 0$.
- ▶ In other words, the nonlinear term $|\nabla h|^2$ has a significant effect on the solution.
- ▶ To prove this, we consider a kind of Fourier decomposition of the solution, and show that high Fourier coefficients have a nonzero contribution, whereas their contribution is zero if $\lambda = 0$.

A very recent development

- ▶ Just a few days ago, roughly three months after we posted our paper, Caravenna, Sun and Zygouras posted a manuscript where they show that the scaling limit exists for all $\lambda \in (0, \lambda_c)$, where λ_c is possibly the correct critical value. (Recall that $\lambda_\varepsilon = \lambda |\log \varepsilon|^{-1/2}$ is the coefficient of $\frac{1}{2} |\nabla h_\varepsilon|^2$.)
- ▶ Moreover, they also show that the scaling limit is Gaussian, and in fact has the same law as the solution of a stochastic heat equation with additive noise, which is not the same as the one obtained by putting $\lambda = 0$.
- ▶ In physics, this is known as the **Edwards–Wilkinson limit**.

Non-trivial scaling limits

- ▶ What happens at criticality?
- ▶ Renormalization group arguments suggest that instead of taking $\lambda = \lambda_c$, if we use $\lambda = \lambda_c +$ correction term, where the correction term is appropriately vanishing as $\varepsilon \rightarrow 0$, it may be possible to construct a **non-Gaussian scaling limit**.
- ▶ Such limits are called **non-trivial** in physics.
- ▶ Getting any non-Gaussian scaling limit for 2D KPZ, either in the above sense, or in the Family–Vicsek scaling, or some other scaling — remains an open problem.