Local KPZ behavior under arbitrary scaling limits

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The KPZ equation

- The Kardar–Parisi–Zhang (KPZ) equation models the growth of a generic randomly growing surface.
- If $f(t,x)$ is the height of a d-dimensional surface at time $t \in \mathbb{R}_{\geq 0}$ and location $x \in \mathbb{R}^d$, the KPZ equation says

$$\partial_t f = \nu \Delta f + \frac{\lambda}{2} |\nabla f|^2 + \sqrt{D}\xi,$$

  where ξ is a random field known as space-time white noise, and ν, λ and D are three strictly positive parameters.
- Formally, space-time white noise is a distribution-valued centered Gaussian random field, with covariance

$$\mathbb{E}(\xi(t,x)\xi(t',x')) = \delta(t-t')\delta^{(d)}(x-x'),$$

  where δ and $\delta^{(d)}$ are the Dirac delta functions on $\mathbb{R}$ and $\mathbb{R}^d$, respectively. (Can be rigorously defined.)

History and literature

- The KPZ equation was introduced by Kardar, Parisi, and Zhang in 1986 as a ‘universal’ model for random surface growth, in the same way that Brownian motion is a universal model for many 1D random processes.
- The study of the KPZ equation and related topics is one of the most active areas in probability today.
- The literature is huge. Will skip over this, due to time constraint. (See in the paper.)
- Let me only briefly mention that KPZ is now well-understood in 1D, but remains mysterious in $d \geq 2$.

The problem of scaling limits

- A fundamental roadblock in constructing nontrivial solutions of the KPZ equation in $d \geq 2$ is that *we do not know how to take scaling limits* of approximate solutions to reach a nontrivial limit.
- Even in 1D, there can be many different scaling limits.
- But in many 1D models, we know at least one way of taking a scaling limit that leads to a nontrivial solution.
- In higher dimensions, the question becomes less tractable.
- Physicists believe that for 2D models, the celebrated Family–Vicsek scaling is the correct one, and leads to a function-valued, rather than distribution-valued, solution of the 2D KPZ equation. (Distribution-valued solutions have been rigorously constructed in $d \geq 2$ in recent years.)
- This has been verified in numerical simulations, but remains out of the reach of rigorous mathematics.

Content of this talk

- I will present a small step towards understanding KPZ in $d \geq 2$ without confronting the issue of constructing scaling limits.
- The approach is based on a framework introduced recently in Chatterjee '21a '21b and Chatterjee & Souganidis '21.
- Since the 'correct' way to scale is still mysterious, the following workaround is proposed.
- Consider a general class of growth models, which contains at least one model of widespread interest.
- Then show that, *irrespective of how we take a scaling limit*, the growth is always *locally* like the KPZ equation, breaking up as the sum of a Laplacian term, a gradient squared term, a noise term, and a residual term that is negligible compared to the other three terms and their sum.
- Surprisingly, this turns out to be doable. The details are in the following slides.

Defining local KPZ growth: Step 1

- Take any $d \geq 1$. Suppose that we have a collection of random functions $\{f_\varepsilon\}_{\varepsilon>0}$ from $\mathbb{Z}_{\geq 0} \times \mathbb{Z}^d$ into $\mathbb{R}$.
- A general 'rescaling' of f_ε is defined as follows.
- Let $\alpha(\varepsilon)$, $\beta(\varepsilon)$ and $\gamma(\varepsilon)$ be positive real numbers depending on ε, such that $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ tend to zero as $\varepsilon \to 0$.
- Based on these coefficients, the rescaled version of f_ε is the function $f^{(\varepsilon)} : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}$ defined as

$$f^{(\varepsilon)}(t, x) := \gamma(\varepsilon) f_\varepsilon(\lceil \alpha(\varepsilon)^{-1} t \rceil, \lceil \beta(\varepsilon)^{-1} x \rceil).$$

- This means we are rescaling space and time so that successive time points are separated by $\alpha(\varepsilon)$ and neighboring points in space are separated by $\beta(\varepsilon)$.
- The factor $\gamma(\varepsilon)$ is just a multiplicative factor meant to ensure that the limit of $f^{(\varepsilon)}$ as $\varepsilon \to 0$ (on some appropriate space of functions or distributions) is nontrivial.

Defining local KPZ growth: Step 2

- Let $A = \{0, \pm e_1, \ldots, \pm e_d\}$ be the set consisting of the origin and its nearest neighbors in $\mathbb{Z}^d$.
- Define the 'local average' of $f^{(\varepsilon)}$ at $(t, x) \in \mathbb{R}_{>0} \times \mathbb{R}^d$ as

$$\overline{f}^{(\varepsilon)}(t, x) := \frac{1}{2d+1} \sum_{a \in A} f^{(\varepsilon)}(t, x + \beta(\varepsilon)a).$$

- Define the 'approximate time derivative' of $f^{(\varepsilon)}$ as

$$\widetilde{\partial}_t f^{(\varepsilon)}(t, x) := \frac{f^{(\varepsilon)}(t + \alpha(\varepsilon), x) - f^{(\varepsilon)}(t, x)}{\alpha(\varepsilon)},$$

- Define the 'approximate Laplacian' as

$$\widetilde{\Delta} f^{(\varepsilon)}(t, x) := (2d+1)\left(\frac{\overline{f}^{(\varepsilon)}(t, x) - f^{(\varepsilon)}(t, x)}{\beta(\varepsilon)^2}\right).$$

Definine local KPZ growth: Step 2 (continued)

- Define the 'approximate squared gradient' as

$$|\widetilde{\nabla} f^{(\varepsilon)}(t,x)|^2 := \frac{1}{2}\sum_{a\in A}\biggl(\frac{f^{(\varepsilon)}(t,x+\beta(\varepsilon)a) - \overline{f}^{(\varepsilon)}(t,x)}{\beta(\varepsilon)}\biggr)^2.$$

- The above definitions are inspired by the fact that if $\alpha(\varepsilon) \to 0$, $\beta(\varepsilon) \to 0$, and $f^{(\varepsilon)}$ converges in some strong sense to a smooth function f as $\varepsilon \to 0$, then the approximate time derivative, the approximate Laplacian, and the approximate squared gradient converge to $\partial_t f$, Δf and $|\nabla f|^2$.
- Of course, we do not expect $f^{(\varepsilon)}$ to converge to a smooth limit in general.

Definition of local KPZ growth

Definition

We will say that $f^{(\varepsilon)}$ has 'local KPZ behavior' as $\varepsilon \to 0$ if for some strictly positive $\nu(\varepsilon)$, $\lambda(\varepsilon)$, $D(\varepsilon)$, and some random maps $\xi^{(\varepsilon)}$ and $R^{(\varepsilon)}$ from $\mathbb{R}_{>0} \times \mathbb{R}^d$ into $\mathbb{R}$, we have that for all $(t, x) \in \mathbb{R}_{>0} \times \mathbb{R}^d$,

$$\widetilde{\partial}_t f^{(\varepsilon)}(t,x) = \nu(\varepsilon)\widetilde{\Delta} f^{(\varepsilon)}(t,x) + \frac{\lambda(\varepsilon)}{2}|\widetilde{\nabla} f^{(\varepsilon)}(t,x)|^2 + \sqrt{D(\varepsilon)}\xi^{(\varepsilon)}(t,x) + R^{(\varepsilon)}(t,x),$$

such that the following hold:

1. *The noise field $\xi^{(\varepsilon)}$ converges in law to white noise on $\mathbb{R}_{>0} \times \mathbb{R}^d$ as $\varepsilon \to 0$.*
2. *The remainder term $R^{(\varepsilon)}(t,x)$ is o_P of the first three terms on the right and their sum, meaning that $R^{(\varepsilon)}(t,x)$ divided by any of the first three terms, or by their sum, tends to zero in probability as $\varepsilon \to 0$.*

Remarks

- It may seem as if $\nu(\varepsilon)$, $\lambda(\varepsilon)$ and $D(\varepsilon)$ should not be allowed to vary with ε, since the coefficients of Δf, $|\nabla f|^2$, and ξ in the KPZ equation are constants.
- However, this is not true. In the KPZ literature, it is understood that the coefficients can be allowed to vary when taking a scaling limit, and even be allowed to tend to zero or blow up to infinity.
- This is especially true in dimensions higher than one. For example, the Family–Vicsek scaling for 2D surfaces requires this.

Motivation

- The motivation for this 'local' definition of KPZ growth comes from a desire to create a universal framework for scaling limits of discrete surface growth, analogous to diffusion approximations of Markov chains.
- Recall that a diffusion process $\{X_t\}_{t\ge 0}$ adapted to a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ is characterized by the 'local growth conditions'

$$\begin{aligned}
&\mathbb{E}(X_{t+h} - X_t|\mathcal{F}_t) = a(X_t)h + o(h),\\
&\mathbb{E}((X_{t+h} - X_t)^2|\mathcal{F}_t) = b(X_t)h + o(h),\\
&\mathbb{E}|X_{t+h} - X_t|^3 = o(h)
\end{aligned}$$

as $h \to 0$, for any fixed t. This gives an informal representation of the stochastic differential equation

$$dX_t = a(X_t)dt + \sqrt{b(X_t)}dB_t$$

without actually defining it rigorously. The purpose of local KPZ growth is to do the same for surface growth.

A class of growing random surfaces

- Recall that $A = \{0, \pm e_1, \ldots, \pm e_d\}$.
- Let $\phi : \mathbb{R}^A \to \mathbb{R}$ be a function.
- Let $\mathbf{z} = \{z_{t,x}\}_{t \in \mathbb{Z}_{>0}, x \in \mathbb{Z}^d}$ be a collection of i.i.d. random variables, which we will refer to as the 'noise field'.
- Given $\varepsilon > 0$, consider a function $f_\varepsilon : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d \to \mathbb{R}$ defined as follows: $f_\varepsilon(0, x) = 0$ for all x, and for each $t \geq 0$,
$$f_\varepsilon(t+1, x) := \phi((f_\varepsilon(t, x+a))_{a \in A}) + \varepsilon z_{t+1,x}.$$
- Imagine $f_\varepsilon(t, x)$ to be the height of a d-dimensional random surface at time t and location x. The above recursion says that the height at time $t+1$ is a function of the heights at x and its neighbors at time t, plus an independent random fluctuation.
- This setup was introduced in Chatterjee '21a, '21b and Chatterjee & Souganidis '21.
- We will later see that the model of directed polymers in a random environment can be put in this framework.

Assumptions about ϕ

- Notation: $1 \in \mathbb{R}^A$ denotes the vector of all 1's. For $u \in \mathbb{R}^A$, $\overline{u}$ denotes the average of the coordinates of u. For $u, v \in \mathbb{R}^A$, we write $u \geq v$ if $u_a \geq v_a$ for each $a \in A$.
- We make the following assumptions about ϕ.
- Equivariance under constant shifts: For all $u \in \mathbb{R}^A$ and $c \in \mathbb{R}$, $\phi(u + c1) = \phi(u) + c$.
- Zero at the origin: $\phi(0) = 0$.
- Monotonicity: $\phi(u) \geq \phi(v)$ whenever $u \geq v$.
- Symmetry: $\phi(u)$ remains unchanged under any permutation of the coordinates of u.
- Regularity: ϕ is C^2 in a neighborhood of the origin.
- Nondegeneracy: $\operatorname{Hess} \phi(0) \neq 0$.
- There is one more crucial assumption, stated in the next slide...

The assumption of strict Edwards–Wilkinson domination

- The Edwards–Wilkinson surface growth model fits into our framework with $\phi(u) = \overline{u}$. We assume that our surface grows at least as fast as the Edwards–Wilkinson surface, meaning that $\phi(u) \geq \overline{u}$ for all u.
- Moreover, we assume that this domination is strict, in the following sense: If $\{u_n\}_{n\geq 1}$ is a sequence such that $\phi(u_n) - \overline{u}_n \to 0$, then $u_n - \overline{u}_n 1 \to 0$.
- This is the assumption of strict Edwards–Wilkinson domination.
- We will later see that the model of directed polymers in a random environment satisfies all of the above assumptions.

Assumptions about the noise field

- We make the following assumptions about the noise field $\mathbf{z}$.
- Zero mean. $\mathbb{E}(z_{t,x}) = 0$.
- Boundedness. There is some constant B such that $|z_{t,x}| \le B$ with probability one.
- Absolute continuity. The law of $z_{t,x}$ is absolutely continuous with respect to Lebesgue measure.

Rescaled surface

- Let f_ε be defined using ϕ and the noise field $\mathbf{z}$, as in the preceding slides.
- Let $\alpha(\varepsilon)$, $\beta(\varepsilon)$ and $\gamma(\varepsilon)$ be any positive real numbers such that $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ tend to zero as $\varepsilon \to 0$.
- Define $f^{(\varepsilon)} : \mathbb{R}_{>0} \times \mathbb{R}^d \to \mathbb{R}$ as

$$f^{(\varepsilon)}(t, x) := \gamma(\varepsilon) f_\varepsilon(\lceil \alpha(\varepsilon)^{-1} t \rceil, \lceil \beta(\varepsilon)^{-1} x \rceil).$$

Main result

Theorem (C., 2021)

Suppose that all the stated assumptions about ϕ and $\mathbf{z}$ are satisfied. Let $f^{(\varepsilon)}$ be the rescaled surface defined in the previous slide, with any $\alpha(\varepsilon)$, $\beta(\varepsilon)$ and $\gamma(\varepsilon)$, the sole condition being that $\alpha(\varepsilon), \beta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then as $\varepsilon \to 0$, $f^{(\varepsilon)}$ has local KPZ behavior, with

$$\nu(\varepsilon) = \frac{\beta(\varepsilon)^2}{(2d+1)\alpha(\varepsilon)}, \quad \lambda(\varepsilon) = \frac{2(q-r)\beta(\varepsilon)^2}{\alpha(\varepsilon)\gamma(\varepsilon)},$$
$$D(\varepsilon) = \frac{\sigma^2\varepsilon^2\beta(\varepsilon)^d\gamma(\varepsilon)^2}{\alpha(\varepsilon)},$$

where σ^2 is the variance of the noise variables, q is the value of the diagonal elements of $\operatorname{Hess}\phi(0)$ *(which are all equal due to the symmetry of ϕ), and r is the value of the off-diagonal elements of* $\operatorname{Hess}\phi(0)$.

Remark

- What if we want ν, λ and D to remain bounded above and below by constants as $\varepsilon \to 0$?
- Recall that up to constants, $\nu \sim \beta^2\alpha^{-1}$, $\lambda \sim \beta^2\alpha^{-1}\gamma^{-1}$, and $D \sim \varepsilon^2\beta^d\gamma^2\alpha^{-1}$.
- So if we want $\nu \sim 1$, we need $\beta^2 \sim \alpha$, which gives $\lambda \sim \gamma^{-1}$.
- Thus, to enforce, $\lambda \sim 1$, we would need $\gamma \sim 1$.
- Finally, to have $D \sim 1$, we use $\beta^2 \sim \alpha$ and $\gamma \sim 1$ to get $\varepsilon^2\beta^{d-2} \sim 1$.
- But since $\beta \to 0$ as $\varepsilon \to 0$, this is impossible unless $d = 1$. Thus, any local KPZ limit in $d \geq 2$ must necessarily have some of the coefficients tending to zero or infinity as $\varepsilon \to 0$.
- When $d = 1$, the above argument shows that there is exactly one way of getting ν, λ and D to be bounded above and below by constants, which is to take $\beta \sim \varepsilon^2$, $\alpha \sim \varepsilon^4$, and $\gamma \sim 1$.
- We will see later that for directed polymers, this is the intermediate disorder regime of Alberts, Khanin & Quastel '14.

Example

- Recall that $A = \{0, \pm e_1, \ldots, \pm e_d\}$.
- Let $\phi : \mathbb{R}^A \to \mathbb{R}$ be defined as
$$\phi(u) := \log\biggl(\frac{1}{2d+1}\sum_{a\in A} e^{u_a}\biggr).$$
- Easy to check: ϕ is equivariant under constant shifts, zero at the origin, monotone, symmetric, C^2, and $\operatorname{Hess}\phi(0) \neq 0$.
- Moreover, $\operatorname{Hess}\phi$ positive semidefinite everywhere, which shows that ϕ is convex.
- Thus, the following lemma shows that ϕ strictly dominates Edwards–Wilkinson growth.

Lemma (C., 2021)

If $\phi : \mathbb{R}^A \to \mathbb{R}$ is equivariant under constant shifts, zero at the origin, monotone, symmetric, C^2 in a neighborhood of the origin, $\operatorname{Hess}\phi(0) \neq 0$, *and ϕ is convex, then ϕ satisfies the strict Edwards–Wilkinson domination condition.*

Directed polymers in random environment

- Let f_ε be the discrete random surface generated using the ϕ displayed in the previous slide and a random field $\mathbf{z} = \{z_{t,x}\}_{t\in\mathbb{Z}_{>0}, x\in\mathbb{Z}^d}$.
- By induction on t, it is easy to show that

$$f_\varepsilon(t,x) = \log\left[\frac{1}{(2d+1)^{t-1}} \sum_{P\in\mathcal{P}_t} \exp\left(\varepsilon \sum_{i=0}^{t-1} z_{t-i,x+p_i}\right)\right],$$

  where $\mathcal{P}_t$ is the set of all $P = (p_0,\ldots,p_{t-1}) \in (\mathbb{Z}^d)^t$ such that $p_0 = 0$ and $|p_i - p_{i-1}| \le 1$ for each i, where $|\cdot|$ is the Euclidean norm.
- This is the log-partition function of the $(d+1)$-dimensional directed polymer model on lazy random walk paths of length $t-1$ at inverse temperature ε, in the random environment $\mathbf{z}$.
- Thus, we arrive at the result displayed in the next slide.

Local KPZ behavior of directed polymers

Theorem (C., 2021)

Consider the random surface generated by the $(d+1)$-dimensional directed polymer model at inverse temperature ε. Then the surface behaves locally like

$$\widetilde{\partial}_t f = \nu \widetilde{\Delta} f + \frac{\lambda}{2}|\widetilde{\nabla} f|^2 + \sqrt{D}\xi + \textit{lower order error term},$$

as we send $\varepsilon \to 0$ and scale space and time in any way we like, where $\widetilde{\partial}_t f$, $\widetilde{\Delta} f$ and $|\widetilde{\nabla} f|^2$ are discrete approximations to the time-derivative, spatial Laplacian, and the squared gradient of f, ξ is an approximation of white noise, and ν, λ, and D are functions of ε and our choice of space-time scaling (that were displayed in an earlier slide).

Intermediate disorder regime

- Alberts, Khanin & Quastel '14 showed that a nontrivial (global) KPZ limit can be obtained for the log-partition function of the $(1+1)$-dimensional directed polymer model by taking the inverse temperature proportional to $n^{-1/4}$, where n is the length of the polymer, and looking at polymers in a spatial window of width $n^{1/2}$. They called this the intermediate disorder regime.
- In our language, the length of the polymer is $\alpha(\varepsilon)^{-1}$, the spatial window has width $\beta(\varepsilon)^{-1}$, and the inverse temperature is ε.
- So the intermediate disorder regime is equivalent to taking $\alpha(\varepsilon) \propto \varepsilon^4$, $\beta(\varepsilon) \propto \varepsilon^2$, and $\gamma(\varepsilon) = \text{constant}$.
- As we saw earlier, this is the only way to have that the coefficients $\nu(\varepsilon)$, $\lambda(\varepsilon)$ and $D(\varepsilon)$ in our equation do not depend on ε.
- Moreover, there is no way to do this when $d \geq 2$.

Proof of the main result: Step 0

- Define the 'local average' of f_ε as
$$\overline{f}_\varepsilon(t,x) := \frac{1}{2d+1}\sum_{a\in A} f_\varepsilon(t,x+a),$$
- If $f_\varepsilon(t,x) \approx f_\varepsilon(t,x+a)$ for all $a \in A$, then by Taylor expansion and using the properties of ϕ, one can show that
$$\begin{aligned}&f_\varepsilon(t+1,x) - f_\varepsilon(t,x)\\ &= \underbrace{\overline{f}_\varepsilon(t,x) - f_\varepsilon(t,x)}_{\text{Laplacian term}} + K\underbrace{\sum_{a\in A}(f_\varepsilon(t,x+a) - \overline{f}_\varepsilon(t,x))^2}_{\text{Gradient squared term}}\\ &\qquad + \varepsilon z_{t+1,x} + \text{a remainder term of smaller order,}\end{aligned}$$
where K is a constant depending on ϕ.
- Thus, to get local KPZ behavior under arbitrary scaling limits, we need to have $f_\varepsilon(t,x+a) \approx f_\varepsilon(t,x)$ for $a \in A$ even if t and x are allowed to vary arbitrarily as $\varepsilon \to 0$.

Proof of the main result: Step 1

- Recall: $f_\varepsilon(0,x) = 0$ for all x, and
$$f_\varepsilon(t+1,x) = \phi((f_\varepsilon(t,x+a))_{a\in A}) + \varepsilon z_{t+1,x},$$
where $A = \{0, \pm e_1, \ldots, \pm e_d\}$.
- Let $\partial_a \phi$ denote the derivative of ϕ in coordinate $a \in A$.
- By equivariance under constant shifts, $\phi(u + c1) = \phi(u) + c$.
- Taking derivative in c at $c = 0$ shows that $\sum_{a\in A} \partial_a \phi(u) = 1$.
- By monotonicity, $\partial_a \phi(u) \geq 0$.
- Thus, at any u, the derivatives of ϕ are nonnegative and sum to 1.

Proof of the main result: Step 2

- Fix a realization of f_ε, and some $t \geq 1$ and $x \in \mathbb{Z}^d$.
- Define a random walk on $\mathbb{Z}^d$ as follows.
- The walk starts at x at time t, and goes backwards in time, until reaching time 0.
- If the walk is at $y \in \mathbb{Z}^d$ at time $s \geq 1$, then at time $s-1$ it moves to $y+a$ with probability $\partial_a \phi((f_\varepsilon(s-1, y+a))_{a\in A})$, for $a \in A$.
- By the observation from the previous slide, these numbers are nonnegative and sum to 1. Therefore, this describes a legitimate random walk on $\mathbb{Z}^d$, moving backwards in time.
- Inductively, one can show that

$$\frac{\partial}{\partial z_{s,y}} f_\varepsilon(t,x) = \varepsilon \mathbb{P}(S_s = y)$$

for any $1 \leq s \leq t$ and $y \in \mathbb{Z}^d$. Note that the probability on the right is conditional probability given f_ε.

Proof of the main result: Step 3

- In particular, for any $1 \leq s \leq t$,

$$\sum_{y \in \mathbb{Z}^d} \left| \frac{\partial}{\partial z_{s,y}} f_\varepsilon(t,x) \right| = \varepsilon.$$

- By this identity and the multivariate mean-value theorem, it follows that if $z_{1,y}$ is replaced by 0 for each y, then the value of $f_\varepsilon(t,x)$ changes by at most $B\varepsilon$, where B is a constant upper bound on the magnitude of the noise variables.
- Let $g_\varepsilon(t,x)$ be the value of $f_\varepsilon(t,x)$ after replacing all $z_{1,y}$ by 0.
- Note that $g_\varepsilon(1,x) = 0$ for each x. Thus, g_ε is just like f_ε, except that instead of starting with an all zero initial condition at time 0, we start with an all zero initial condition at time 1.
- Thus, $g_\varepsilon(t+1,x)$ has the same law as $f_\varepsilon(t,x)$.
- This gives us:

$$\mathbb{E}(f_\varepsilon(t+1,x) - f_\varepsilon(t,x)) = \mathbb{E}(f_\varepsilon(t+1,x) - g_\varepsilon(t+1,x)) \leq B\varepsilon.$$

Proof of the main result: Step 3

- Since the law of $f_\varepsilon(t,x)$ is the same for all x, this gives

$$\mathbb{E}(f_\varepsilon(t+1,x) - \overline{f}_\varepsilon(t,x)) = \mathbb{E}(f_\varepsilon(t+1,x) - f_\varepsilon(t,x)) \le B\varepsilon.$$

- Since the noise variables have mean zero,

$$\begin{aligned}\mathbb{E}(f_\varepsilon(t+1,x)) &= \mathbb{E}(\phi((f_\varepsilon(t,x+a))_{a\in A}) + \varepsilon z_{t+1,x}) \\ &= \mathbb{E}(\phi((f_\varepsilon(t,x+a))_{a\in A})).\end{aligned}$$

- Combining, and using equivariance of ϕ under constant shifts, we get $\mathbb{E}(\phi(q_\varepsilon(t,x))) \le B\varepsilon$, where

$$q_\varepsilon(t,x) := (f_\varepsilon(t,x+a) - \overline{f}_\varepsilon(t,x))_{a\in A}.$$

Proof of the main result: Step 4

- Note that the vector $q_\varepsilon(t,x)$ belongs to the hyperplane $H := \{u \in \mathbb{R}^A : \overline{u} = 0\}$.
- By Edwards–Wilkinson domination, ϕ is nonnegative everywhere on this hyperplane. Thus, for any $\eta > 0$,

$$\mathbb{P}(\phi(q_\varepsilon(t,x)) > \eta) \le \frac{\mathbb{E}(\phi(q_\varepsilon(t,x)))}{\eta} \le \frac{B\varepsilon}{\eta}.$$

- By strict EW domination, $\phi(u_n) \to 0$ implies $u_n \to 0$ on H.
- This implies that for any $\delta > 0$, there exists $\eta(\delta) > 0$ such that $u \in H$ and $\phi(u) \le \eta(\delta)$ implies $|u| \le \delta$.
- Thus,

$$\mathbb{P}(|q_\varepsilon(t,x)| > \delta) \le \mathbb{P}(\phi(q_\varepsilon(t,x)) > \eta(\delta)) \le \frac{B\varepsilon}{\eta(\delta)}.$$

- Note: The bound has no dependence on t and x.
- Thus, if $\varepsilon \to 0$ and $t_\varepsilon, x_\varepsilon$ vary arbitrarily with ε, the above inequality shows that $|q_\varepsilon(t_\varepsilon, x_\varepsilon)| \to 0$ in probability.

Completing the proof

- Since $q_\varepsilon(t, x) = (f_\varepsilon(t, x + a) - \overline{f}_\varepsilon(t, x))_{a \in A}$, this already shows that we can apply Taylor expansion even if t and x vary arbitrarily as $\varepsilon \to 0$.
- Some more work is needed to establish that the remainder term is negligible compared to the other terms. This requires the assumption that $\operatorname{Hess} \phi(0) \neq 0$. I will omit this part.