Wilson loops in Ising lattice gauge theory

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Definition of lattice gauge theories

- Lattice gauge theories are discrete Euclidean quantum field theories.
- Let d = dimension of spacetime, and G be a matrix Lie group. (Most important: $d = 4$ and $G = SU(2)$ or $SU(3)$.)
- The lattice gauge theory with gauge group G on a finite set $\Lambda \subseteq \mathbb{Z}^d$ is defined as follows.
- Suppose that for any two adjacent vertices $x, y \in \Lambda$, we have a group element $U(x, y) \in G$, with $U(y, x) = U(x, y)^{-1}$.
- Let $G(\Lambda)$ denote the set of all such configurations.
- A square bounded by four edges is called a plaquette. Let $P(\Lambda)$ denote the set of all plaquettes in Λ.

Definition of lattice gauge theory contd.

- For a plaquette $p \in P(\Lambda)$ with vertices x_1, x_2, x_3, x_4 in anti-clockwise order, and a configuration $U \in G(\Lambda)$, define
$$U_p := U(x_1, x_2)U(x_2, x_3)U(x_3, x_4)U(x_4, x_1).$$
- The Wilson action of U is defined as
$$S_{\mathrm{W}}(U) := \sum_{p \in P(\Lambda)} \mathrm{Re}(\mathrm{Tr}(I - U_p)).$$
- Let σ_Λ be the product Haar measure on $G(\Lambda)$.
- Given $\beta > 0$, let $\mu_{\Lambda,\beta}$ be the probability measure on $G(\Lambda)$ defined as
$$d\mu_{\Lambda,\beta}(U) := \frac{1}{Z} e^{-\beta S_{\mathrm{W}}(U)} d\sigma_\Lambda(U),$$
where Z is the normalizing constant.
- This probability measure is called the lattice gauge theory on Λ for the gauge group G, with inverse coupling strength β.
- An infinite volume limit is obtained by taking $\Lambda \uparrow \mathbb{Z}^d$.

Open questions about lattice gauge theories

- Although a considerable number of deep mathematical results about lattice gauge theories have been proved, the most important questions remain open.
- In particular:
  - The problem of constructing a continuum scaling limit as the lattice spacing $\to 0$ and $\beta \to \infty$ simultaneously is the main step in the open problem of Yang–Mills existence.
  - The problem of showing exponential decay of correlations at large β is the main step in the open problem of Yang–Mills mass gap.
  - The problem of understanding the expectations of Wilson loop variables at large β is the main step in the open problem of quark confinement.
- This talk is about a tiny little advance for the third problem.
- In the interest of time, I will not attempt to survey the literature. See my Yang–Mills survey on arXiv if you are interested.

Wilson loops

- Consider a lattice gauge theory on $\mathbb{Z}^d$ with gauge group G.
- Let U be a random configuration of group elements attached to edges, drawn from the probability measure defined by this theory.
- Given a loop γ with directed edges $e_1, \ldots, e_m$, the Wilson loop variable W_γ is defined as

$$W_\gamma := \mathrm{Re}(\mathrm{Tr}(U(e_1)U(e_2)\cdots U(e_m))).$$

- The expected value of W_γ is denoted by $\langle W_\gamma \rangle$.
- Wilson loop expectations are crucial for understanding confinement of quarks, and also for attempts at constructing continuum scaling limits.
- Well-understood in $d = 2$. However, $d = 4$ is the important dimension.

What is known in $d \geq 3$

- General results:
  - $C_1 e^{-C_2 \text{area}(\gamma)} \leq \langle W_\gamma \rangle \leq C_3 e^{-C_4 \text{perimeter}(\gamma)}$, where area($\gamma$) is the minimum surface area enclosed by γ, perimeter(γ) is the perimeter of γ, and C_i are constants that do not depend on γ, but may depend on β, d, and the gauge group G.
  - The lower bound is a result of Seiler (1978) and the upper bound was proved by Simon and Yaffe (1982).
  - When β is small, the lower bound was shown to be sharp by Osterwalder and Seiler (1978), in the sense that an upper bound of the form $C_5 e^{-C_6 \text{area}(\gamma)}$ was proved.
  - A series expansion for $\langle W_\gamma \rangle$ at small β in terms of a lattice string theory was established in the works of Chatterjee (2015) and Chatterjee and Jafarov (2016).
- Special results:
  - When $d = 3$ and $G = U(1)$, the lower bound was shown to be sharp for all β by Göpfert and Mack (1982).
  - When $d = 4$ and $G = U(1)$, the upper bound was shown to be sharp at large β by Guth (1980) and Fröhlich and Spencer (1982).

Main unsolved questions in $d \geq 3$

- From the list of available results, we see that matching upper and lower bounds for $\langle W_\gamma \rangle$ are unknown at large β, expect in the special case of $G = U(1)$.
- For constructing continuum limits of non-Abelian gauge theories (Yang–Mills existence), it does not suffice to understand $\langle W_\gamma \rangle$ up to matching constants in the exponent — we actually need a very precise understanding of $\langle W_\gamma \rangle$ as $\beta \to \infty$, and in particular we need to know the exact constant in the exponent.
- I will now present such a result in $d = 4$ for the simplest gauge group, namely, $\mathbb{Z}_2$ ($= \{-1, 1\}$ with multiplication). This is known as 4D Ising lattice gauge theory.

Wilson loop expectation in 4D $\mathbb{Z}_2$ lattice gauge theory

- Call an edge $e \in \gamma$ a corner edge if there is some other edge $e' \in \gamma$ such that e and e' share a common plaquette.
- For example, a rectangular loop with length and width greater than one has exactly eight corner edges.

Theorem (C., 2018)

Consider lattice gauge theory with $d = 4$ and $G = \mathbb{Z}_2$. There exists $\beta_0 > 0$ such that the following holds when $\beta \geq \beta_0$. Let γ be a non-self-intersecting loop. Let ℓ be the number of edges in γ and let ℓ_0 be the number of corner edges of γ. Then

$$|\langle W_\gamma \rangle - e^{-2\ell e^{-12\beta}}| \leq C_1 \left(e^{-2\beta} + \sqrt{\frac{\ell_0}{\ell}} \right)^{C_2},$$

where C_1 and C_2 are two positive universal constants.

(In other words, if $\beta \gg 1$ and $\ell_0 \ll \ell$, and $\ell = \alpha e^{12\beta}$, then $\langle W_\gamma \rangle \approx e^{-2\alpha}$.)

Open questions

- The result is a small mathematical step towards understanding the precise behavior of Wilson loop expectations at large β.
- The proof technique can possibly be extended to general finite Abelian gauge groups.
- Extension to infinite Abelian groups, such as $U(1)$, would be very interesting.
- Extension to non-Abelian groups would be a really important breakthrough. Finite non-Abelian groups will probably be easier than infinite.

Plan for the remainder of this talk

- In the rest of this talk, I will outline the main steps in the proof of the theorem.
- The key tools are:
  - Discrete exterior calculus — discrete Poincaré lemma and discrete Hodge duality.
  - A duality relation for Wilson loop expectations in 4D Ising lattice gauge theory. Derived using the character expansion for $\mathbb{Z}_2$, possibly allowing generalizations to other Abelian groups.
  - Techniques for proving correlation decay.
  - A resampling trick and Poisson approximation.

4D Ising lattice gauge theory

- For convenience, let us recall the definition of the theory.
- A configuration σ is an assignment of spins $\sigma_e \in \{-1, 1\}$ to edges e of $\mathbb{Z}^4$.
- If p is a plaquette with edges e_1, e_2, e_3, e_4, define

$$\sigma_p = \sigma_{e_1}\sigma_{e_2}\sigma_{e_3}\sigma_{e_4}.$$

(Note that we do not have to take σ_e^{-1} since $x = x^{-1}$ in $\mathbb{Z}_2$.)
- Let Λ be a finite subset of $\mathbb{Z}^4$. Let $E(\Lambda)$ be the set of edges of Λ and $P(\Lambda)$ be the set of plaquettes of Λ.
- The probability measure defined by Ising lattice gauge theory on $\{-1, 1\}^{E(\Lambda)}$ at inverse coupling strength β is simply

$$\mu_\beta(\sigma) = \frac{1}{Z}\exp\biggl(\beta \sum_{p\in P(\Lambda)} \sigma_p\biggr),$$

where Z is the normalizing constant.

The cell complex of $\mathbb{Z}^4$

- $\mathbb{Z}^4$ can be viewed as a union of 4-dimensional unit cubes.
- These cubes are also called 4-cells. Each 4-cell has a number of faces of dimensions 0, 1, 2 and 3. The k-dimensional faces are called k-cells.
- In particular, a 0-cell is a vertex, a 1-cell is an edge, and a 2-cell is a plaquette.
- The collection of all these k-cells, for $0 \leq k \leq 4$, is called the cell complex of $\mathbb{Z}^4$.

The dual lattice

- The centers of the 4-cells of $\mathbb{Z}^4$ form another copy of $\mathbb{Z}^4$, that is known as the dual lattice of $\mathbb{Z}^4$.
- The k-cells of the primal lattice are in a one-to-one correspondence with the $(4-k)$-cells of the dual lattice.
- In particular, the plaquettes of the primal lattice correspond to the plaquettes of the dual lattice. This is a special feature of 4D, which will be very important for us.
- The dual of a plaquette p is a plaquette in the dual lattice, which will be denoted by $*p$.

Surfaces in $\mathbb{Z}^4$

- Any collection of P plaquettes will be called a surface.
- The boundary ∂P of a surface P is the set of edges that belong to an odd number of plaquettes in P.
- If the boundary is empty, the surface will be called closed.
- If P is a surface, the set of dual plaquettes will be called the dual surface, and will be denoted by $*P$.
- Important lemma, derived using the discrete Poincaré lemma: *For any closed loop γ, there is at least one surface P with $\gamma = \partial P$.*
- Consequence:

$$\boxed{W_\gamma = \prod_{p \in P} \sigma_p.}$$

- Remark: There should be an analogous formula for any Abelian theory. Does not seem to extend easily to non-Abelian theories.

A duality for Wilson loop expectations

- Let $\langle \cdot \rangle_\beta$ denote expectation in Ising lattice gauge theory at inverse coupling strength β and free boundary condition (no restriction on boundary spins).
- Let us also define Ising lattice gauge theory on the dual lattice, and let $\langle \cdot \rangle^*_\beta$ denote expectation in this dual theory at inverse coupling strength β and zero boundary condition (boundary spins are all 1 — since 1 is the zero of $\mathbb{Z}_2$).

Theorem (C., 2018)

Let P be any finite surface and $\lambda := -\frac{1}{2}\log\tanh\beta$. Then

$$\left\langle \prod_{p\in P} \sigma_p \right\rangle_\beta = \left\langle e^{-2\lambda \sum_{p \in *P} \sigma_p} \right\rangle^*_\lambda .$$

(Proof uses discrete exterior calculus and the character expansion for $\mathbb{Z}_2$. Possibly generalizable to other Abelian groups.)

A key consequence of the duality

- As $\beta \to \infty$, $\lambda \to 0$.
- This allows us to study Wilson loop expectations at large β using strong coupling (small β) techniques in the dual lattice.
- A basic feature of strong coupling is that correlations decay exponentially. This is proved using standard technology, pioneered by Dobrushin.
- The correlation decay can be transferred to weak coupling (large β) using the duality theorem. At weak coupling the σ_p's have exponential decay of correlations, and not the σ_e's.
- The following lemma is proved using correlation decay at weak coupling. Probably generalizable to other Abelian groups. Not clear if there is a non-Abelian generalzation (more later).

Lemma

Let P be any finite surface. There is a constant C_P depending only on P such that under μ_β, the probability of the event that $\sigma_p = -1$ for all $p \in P$ is bounded above by $C_P e^{-2\beta|P|}$.

Vortices

- Given a configuration of spins σ, a surface P is called a vortex if $*P$ is closed and connected, and $\sigma_p = -1$ for each $p \in P$.
- The smallest vortex is of size 6, and consists of the set of 6 plaquettes containing a given edge e. Its dual consists of the set of 6 plaquettes forming the boundary of the 3-cell which is the dual of e. Such a vortex will be called a minimal vortex, and will be denoted by $P(e)$.

Two lemmas about vortices

The following lemmas are proved using discrete exterior calculus.

Lemma

Given any configuration σ, the set of all plaquettes p with $\sigma_p = -1$ is a disjoint union of vortices.

Lemma

If Q is a surface and P is a vortex contained in a 4D box that does not intersect the boundary of Q, then $|P \cap Q|$ is even.

Completing the proof of the main theorem

- Let γ be a non-self-intersecting loop of length ℓ. Recall that we have to show that $\langle W_\gamma \rangle \approx e^{-2\ell e^{-12\beta}}$ when β is large and the fraction of corner edges is small.
- If $\ell \ll e^{12\beta}$, we have to show that $\langle W_\gamma \rangle \approx 1$, and if $\ell \gg e^{12\beta}$, we have to show that $\langle W_\gamma \rangle \approx 0$. Let us not worry about these special cases, and only consider the case where ℓ is comparable to $e^{12\beta}$.
- Suppose $\ell = \alpha e^{12\beta}$. We have to show that $\langle W_\gamma \rangle \approx e^{-2\alpha}$.

A first simplification

- Recall that there is some surface P such that $\gamma = \partial P$.
- Recall also that $W_\gamma = \prod_{p \in P} \sigma_p$.
- Thus, $W_\gamma = (-1)^N$, where $N = |\{p \in P : \sigma_p = -1\}|$.
- But we know that $\{p : \sigma_p = -1\}$ is a disjoint union of vortices.
- We also know that a vortex 'away from' γ has an even-sized intersection with P.
- Thus, $W_\gamma = (-1)^{N_1}$, where N_1 is the number of plauettes of P belonging to vortices that have some part 'near' γ.

Negative plaquettes in P

- It can be arranged that P is contained in a cubical box B of width ℓ.
- Recall that for any surface Q,

$$\mu_\beta\{\sigma_p = -1 \ \ \forall p \in Q\} \le C_Q e^{-2\beta|Q|}.$$

- From this and the fact that duals of vortices are connected, it follows that

$$\mu_\beta\{\text{Some vortex of size} \ge 25 \text{ intersects } B\} \le C\ell^4 e^{-50\beta}$$

- But $\ell = \alpha e^{12\beta}$, and so $\ell^4 = \alpha^4 e^{48\beta}$. Thus, when β is large, the above bound is small.
- So, with high probability, only vortices of size ≤ 24 can intersect P.
- Combining this with our previous observation that only vortices coming near the boundary matter, we see that $W_\gamma = (-1)^{N_2}$, where N_2 is the number of plaquettes in P that are contained in vortices that are within distance 24 from γ.

Vortices near the boundary

- Let R be the set of plaquettes within distance 24 from γ, so that $|R| \le C\ell$.
- Then again by our lemma,

$$\mu_\beta\{\text{Some vortex of size} \ge 7 \text{ intersects } R\} \le C\ell e^{-14\beta}.$$

- But since $\ell = \alpha e^{12\beta}$, this probability is small when β is large.
- But we know that a vortex of size ≤ 6 must be a minimal vortex, that is, the set $P(e)$ of all plaquettes containing some given edge e.
- Thus, with high probability, only such vortices occur in R.
- If $P(e)$ is such a vortex, but $e \notin \gamma$, then $|P(e) \cap P|$ is even because e is not a boundary edge and hence is contained in an even number of plaquettes of P. On the other hand, if $e \in \gamma$, then $|P(e) \cap P|$ is odd.
- Thus, with high probability, $W_\gamma = (-1)^{N_3}$, where N_3 is the number of $e \in \gamma$ such that $P(e)$ is a vortex.

Final step: A resampling trick

- Let γ_1 be the set of non-corner edges in γ. Since the proportion of corner edges is small, we can further show that with high probability, $W_\gamma = (-1)^{N_4}$, where N_4 is the number of $e \in \gamma_1$ such that $P(e)$ is a vortex.
- A key property of γ_1 is that conditional on $(\sigma_e)_{e \notin \gamma_1}$, the spins $(\sigma_e)_{e \in \gamma_1}$ are independent.
- Thus, if we resample these spins independently from their conditional distributions, we get a new configuration σ' with the same law as σ.
- Take any edge $e \in \gamma_1$. If $\sigma_p = 1$ for all $p \in P(e)$, then a simple calculation shows that $P(e)$ is a vortex in the new configuration σ' with probability $\approx e^{-12\beta}$.
- Thus, if we can show that almost all $e \in \gamma_1$ satisfy the above property, then in the configuration σ', N_4 is approximately a Poisson r.v. with mean $\ell e^{-12\beta} = \alpha$. But if $X \sim Poi(\alpha)$, then $\mathbb{E}(-1)^X = e^{-2\alpha}$, so this would complete the proof.

Wrapping up

- We have to show that almost all $e \in \gamma_1$ have the property that $\sigma_p = 1$ for all $p \in P(e)$.
- Note that for any plaquette p, the event $\sigma_p = -1$ implies that p is contained in a vortex, which must be of size at least 6. Since a vortex is connected, this implies that

$$\mu_\beta\{\sigma_p = -1\} \le Ce^{-12\beta}.$$

- Thus, the expected number of edges in γ_1 violating the above property is bounded by $C\ell e^{-12\beta} = C\alpha$. This completes the sketch of the proof.

A remark about dualities in non-Abelian theories

- I presented a duality for Ising lattice gauge theory that connects the strong and weak coupling regimes.
- Such dualities are known as strong-weak dualities.
- For non-Abelian theories, there is a conjectured set of such dualities, known as the Montonen–Olive dualities.
- In 2007, Kapustin and Witten suggested that the Montonen–Olive dualities are in fact equivalent to the geometric Langlands correspondence.
- Of course, no one knows how to prove anything about any of this.