Constructing a solution of the 2D KPZ equation

Infosys-ICTS Ramanujan Lectures: Lecture 3

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Random surfaces

- In modern probability theory, we have a vast and deep understanding of randomly moving points.
- Most basic examples: simple random walks and Brownian motion. More complex examples: dependent random walks, diffusion processes.
- But we are only beginning to understand randomly moving objects of dimension greater than zero. These are generally called random surfaces, although a 'surface' should have dimension 2, strictly speaking.

Scaling limits

- A key objective in probability theory is to show that discrete processes have continuous scaling limits.
- Roughly speaking, we consider the discrete process on a scaled lattice like $\epsilon\mathbb{Z}^d$, and look at its limiting behavior as $\epsilon \to 0$.
- Often, these limits are universal, in the sense that many different discrete processes have the same scaling limit.
- The primary example is convergence of various discrete random walks to Brownian motion.
- Another example is Schramm–Loewner evolution (SLE). Many types of self-avoiding random walks and other complicated discrete processes have been shown to converge to SLE in the scaling limit, leading to solutions of longstanding problems.

Scaling limits of random surfaces

- Just like Brownian motion and SLE are universal scaling limits for a wide variety of discrete random processes, there is supposed to be a universal limit for growing discrete random surfaces.
- Here 'surface' may be d-dimensional for any positive integer d.
- This limit process is described by the $(d+1)$-dimensional Kardar–Parisi–Zhang (KPZ) equation.
- However, our knowledge about convergence to KPZ is much more limited than our understanding of convergence to Brownian motion or SLE.
- Rigorous proofs of convergence exists only for a few classes of one-dimensional surface growth process. (Huge literature.)
- For $d \geq 2$, even the limit process (i.e. the solution of the KPZ equation) is not well-understood. Questions of convergence can be investigated only after this is settled.

Goal of this talk

- In this talk, I will present the first rigorous construction of a solution of the 2D KPZ equation.
- The solution is obtained by a particular renormalization scheme. It is possible — indeed, probable — that other solutions may be obtained by other schemes.

The $(d+1)$-dimensional Kardar–Parisi–Zhang (KPZ) equation

- Describes a growing random surface $h(t,x)$ ($t \geq 0$, $x \in \mathbb{R}^d$):

$$\boxed{\partial_t h = \nu \Delta h + \frac{\lambda}{2}|\nabla h|^2 + \sqrt{D}\dot{W}}$$

where ν, λ, D are positive constants and $\dot{W}$ is standard space-time white noise.
- Formally, $\dot{W}$ is a centered Gaussian field satisfying

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(t',x')) = \delta(t-t')\delta(x-x').$$

- Rigorously, $\dot{W}$ is a random distribution with the property that for any smooth f with compact support, the integral

$$\int f(t,x)\dot{W}(t,x)dtdx$$

is a Gaussian random variable with mean zero and variance $\int f(t,x)^2 dtdx$.

Remarks

- The KPZ equation was introduced by Kardar, Parisi and Zhang in 1986.
- As stated, it is not even clear that the equation is mathematically meaningful.
- One key difficulty: ∇h is expected to be a random distribution, and therefore $|\nabla h|^2$ is undefined.

The case $d = 1$

- Most of the literature is about $d = 1$.
- A number of discrete models have been shown to converge to the $(1+1)$-dimensional KPZ equation in the scaling limit, justifying its claim to universality.
- The Cole–Hopf solution of the 1D KPZ equation is obtained as $h = (2\nu/\lambda)\log u$, where u solves the stochastic heat equation (SHE) with multiplicative noise:

$$\partial_t u = \nu \partial_x^2 u + (\lambda\sqrt{D}/2\nu)u\dot{W}.$$

- This is a solution only 'in principle', because it is obtained using Itô's formula, and Itô's formula is not applicable here.
- A direct approach to solving the KPZ equation in $d = 1$ was provided by Hairer's theory of regularity structures.
- Other direct approaches: paracontrolled distributions by Gubinelli, Imkeller and Perkowski, energy solutions by Gonçalves and Jara, renormalization approach by Kupiainen.

Problem with the Cole–Hopf solution in $d \geq 2$

- The solutions to the multiplicative SHE are now well-understood in all dimensions.
- The case $d \geq 3$ was analyzed by Mukherjee, Shamov and Zeitouni (2016), and the case $d = 2$ by Caravenna, Sun and Zygouras (2017), and also by Feng (2016), following early contributions of Bertini and Cancrini (1998).
- If u is a solution to the $(d+1)$-dimensional multiplicative SHE, then $h = (2\nu/\lambda)\log u$ is a formal solution of the KPZ equation in any dimension d.
- However, when $d \geq 2$, u is a distribution rather than a function. So it is not clear how to define $\log u$.

Special difficulty in $d = 2$

- The $d = 1$ case has a special advantage: In the language of PDEs, it is subcritical, and in the language of renormalization, it is ultraviolet superrenormalizable.
- All methods for solving the KPZ equation in $d = 1$ depend crucially on the above feature.
- What this means is that there is a way to rescale space and time such that the coefficient of $|\nabla h|^2$ can be made as small as we like while keeping the other two coefficients fixed.
- This is also possible in $d \geq 3$, with a different rescaling. The system is called infrared superrenormalizable. Recently analyzed by Magnen and Unterberger (2018).
- The case $d = 2$ is critical, in PDE language. In the language of renormalization, it is not superrenormalizable.
- To summarize, neither the Cole–Hopf solution nor the direct approaches such as regularity structures are expected to work in $d = 2$. In this sense, $d = 2$ is the hardest dimension.

Mollified equation

- Approaches to solving the KPZ equation usually begin in the following way.
- Let $\dot{W}_\varepsilon$ be the white noise $\dot{W}$ mollified by convolving with a spatial mollifier ρ^ε, where $\rho^\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$, and ρ is a compactly supported smooth function.
- Formally,

$$\dot{W}^\varepsilon(t,x) = \int \dot{W}(t,y)\rho^\varepsilon(x-y)dy.$$

- Consider the mollified equation

$$\partial_t h_\varepsilon = \nu\Delta h_\varepsilon + \frac{\lambda}{2}|\nabla h_\varepsilon|^2 + \sqrt{D}\dot{W}_\varepsilon.$$

- Usually, this equation is solvable in a traditional sense.
- The question is, what happens as $\varepsilon \to 0$?

Renormalization

- The renormalization viewpoint: The parameters ν, λ and D need not be fixed parameters; they are free to depend on ε. The main quest is to vary them in such a way so as to get a meaningful/interesting limit object as $\varepsilon \to 0$.
- Additionally, for the KPZ equation, one may need to subtract off a constant C_ε depending on ε.
- In the context of making numerical predictions, "meaningful/interesting" means "yields numbers that match experiment".

Renormalized scaling limit

- To summarize, we need to consider the equation

$$\partial_t h_\varepsilon = \nu_\varepsilon \Delta h_\varepsilon + \frac{\lambda_\varepsilon}{2}|\nabla h_\varepsilon|^2 + \sqrt{D_\varepsilon}\dot{W}_\varepsilon - C_\varepsilon,$$

  where ν_ε, λ_ε, D_ε and C_ε are constants depending on ε defined in such a way that the solution h_ε converges to some interesting limit as $\varepsilon \to 0$.
- Note that there may be multiple ways to vary ν_ε, λ_ε, D_ε and C_ε with ε that give interesting limits.
- When $d = 1$, it turns out that an interesting limit can be obtained if ν, λ and D are kept fixed and C_ε is made to blow up appropriately as $\varepsilon \to 0$ (so that h_ε always has mean zero).
- In fact this is the Cole–Hopf solution in $d = 1$.
- In $d \geq 3$, Magnen and Unterberger (2018) scaled λ_ε like $\varepsilon^{(d-2)/2}$, blew up C_ε appropriately, and kept ν and D fixed to get a Gaussian scaling limit.

Our result

Theorem (C. and Dunlap, 2018)

Consider the mollified KPZ equation on a 2D torus. Suppose that we keep ν and D fixed, blow up C_ε appropriately, and set $\lambda_\varepsilon = \lambda|\log\varepsilon|^{-1/2}$ for some $\lambda > 0$. That is, consider

$$\partial_t h_\varepsilon = \nu\Delta h_\varepsilon + \frac{\lambda}{2\sqrt{\log(1/\varepsilon)}}|\nabla h_\varepsilon|^2 + \sqrt{D}\dot{W}_\varepsilon - C_\varepsilon.$$

If λ is small enough, the solution h_ε converges in law to a limiting random distribution along some sequence of $\varepsilon \to 0$. Moreover, this limit is not the same as the one obtained by simply putting $\lambda = 0$.

The Family–Vicsek scaling

- The above theorem is the first rigorous result about a scaling limit for 2D KPZ.
- However, our renormalization of parameters is probably not the only possible renormalization that gives an interesting limit.
- Numerical simulations suggest that it may also be possible to obtain a function-valued scaling limit by taking $\nu \sim \varepsilon^{2-z}$, $\lambda \sim \varepsilon^{2-z-\alpha}$, and $D \sim \varepsilon^{2+2\alpha-z}$ for certain exponents α and z. This is known as the Family–Vicsek scaling.
- Scaling arguments suggest that $\alpha + z = 2$. If we assume this, then we obtain the scaling $\nu \sim \varepsilon^{\alpha}$, $\lambda \sim 1$, and $D \sim \varepsilon^{3\alpha}$.
- This amounts to considering the KPZ equation with *fixed* values of the parameters, and considering the solution multiplied by ε^{α} on a short time scale $t \sim \varepsilon^{\alpha}$.
- There is no consensus about the value of α. Mathematically, the problem is wide open.

Tightness

- Our theorem is essentially a tightness result.
- We prove tightness of $\{h_\varepsilon\}_{\varepsilon>0}$ on an appropriate space of distributions.
- This is a kind of Besov space, used previously by Hairer and others in the study of KPZ and other systems.
- A convenient criterion for tightness on these spaces was recently provided by Furlan and Mourrat (2017), which we use.

Proof sketch

- For simplicity, let me just sketch the proof of tightness for the family of random variables

$$X_\varepsilon := \int h_\varepsilon \phi,$$

where ϕ is some compactly supported smooth function on the torus.
- Recall that we have adjusted the renormalization constant C_ε suitably, so that $\mathbb{E}(h_\varepsilon(t, x)) = 0$ for all t, x. Thus, $\mathbb{E}(X_\varepsilon) = 0$.
- So it suffices to prove that $\sup_{0<\varepsilon<1} \mathbb{E}(X_\varepsilon^2) < \infty$.

Proving tightness

- The first step is to write h_ε as $\log u_\varepsilon$, where u_ε is the solution of the mollified multiplicative SHE.
- Thus,
$$X_\varepsilon = \int \phi \log u_\varepsilon.$$
- u_ε has a representation in terms of a Feynman–Kac formula.
- Due to this explicit representation, we can use Malliavin calculus to calculate
$$\frac{\partial}{\partial \lambda}\mathbb{E}(X_\varepsilon^2).$$
- If we can show that the above derivative is uniformly bounded when $0 < \varepsilon < 1$ and $0 \le \lambda \le \lambda_0$ for some $\lambda_0 > 0$, and also show that $\sup_{0<\varepsilon<1} \mathbb{E}(X_\varepsilon^2) < \infty$ when $\lambda = 0$, then we will reach our objective of showing that $\sup_{0<\varepsilon<1} \mathbb{E}(X_\varepsilon^2) < \infty$ when $0 \le \lambda \le \lambda_0$.

Recursion

- Unfortunately, the computation of $\partial\mathbb{E}(X_\varepsilon^2)/\partial\lambda$ yields an expectation of the intersection time of two random paths chosen according to a continuum polymer measure, which is Wiener measure with a random tilt.
- We do not know how to calculate expectations of such intersection times.
- But when $\lambda = 0$, the polymer measure reduces to just 2D Wiener measure. We know how to calculate expectations of Brownian intersection times.
- Thus, we can again take derivative with respect to λ, and hope to get a bound for $\partial^2\mathbb{E}(X_\varepsilon^2)/\partial\lambda^2$.
- But this yields an expectation of a polynomial in the pairwise intersection times of four random paths from a polymer measure.
- One can keep doing this, but it just gets more and more complicated. In k steps, complexity grows like e^{Ck^2}.

Solving the recursion

- To solve the recursion problem, the first step is to reparametrize λ as $\sqrt{\beta}$, and take derivatives with respect to β. Fixing ε, let $g(\beta) := \mathbb{E}(X_\varepsilon^2)$.
- We want to control $g(\beta)$ by controlling $g'(\beta)$, and control $g'(\beta)$ by controlling $g''(\beta)$, and so on.
- The problem is that these derivatives become hopelessly complicated.
- So we find a way around this problem:
  - Set $g_0 = g$. Instead of bounding g', find a nicer function g_1 such that $|g'(\beta)| \le g_1(\beta)$.
  - To bound g_1, we seek to bound g_1'. So we find a nicer function g_2 such that $|g_1'(\beta)| \le g_2(\beta)$.
  - We keep going like this, and end up with a sequence of functions $g_1, g_2, \ldots$ such that $|g_k'(\beta)| \le g_{k+1}(\beta)$ for each k.
  - The nice feature of this sequence is that the complexity grows like e^{Ck}.

Completing the proof

- For each k, $g_k(0)$ can be obtained by a Brownian computation.
- Finally,

$$|g(\beta)| \le \sum_{k=0}^{\infty} \frac{\beta^k}{k!} |g_k(0)|.$$

- Since the complexity of g_k increases only exponentially, and the kth moment of intersection times grow like $k!$, we can bound $|g_k(0)| \le C^k k!$ for some C.
- Thus, for β sufficiently small, we get a finite bound.
- Moreover, it turns out that the bound does not blow up as $\varepsilon \to 0$.

Contribution of the nonlinearity

- The last sentence in our theorem asserts that the scaling limit we obtain is not the same as the one obtained by putting $\lambda = 0$.
- In other words, the nonlinear term $|\nabla h|^2$ has a significant effect on the solution.
- To prove this, we consider a kind of Fourier decomposition of the solution, and show that high Fourier coefficients have a nonzero contribution, whereas their contribution is zero if $\lambda = 0$.

Very recent developments

- We posted our paper in September 2018.
- In December 2018, there was a flurry of new advances:
  - Caravenna, Sun and Zygouras posted a manuscript where they show that the scaling limit exists for all $\lambda \in (0, \lambda_c)$, where λ_c is possibly the correct critical value. (Recall that $\lambda_\varepsilon = \lambda |\log \varepsilon|^{-1/2}$ is the coefficient of $\frac{1}{2}|\nabla h_\varepsilon|^2$.)
  - Moreover, they also show that the scaling limit is Gaussian, and in fact has the same law as the solution of a stochastic heat equation with additive noise, which is not the same as the one obtained by putting $\lambda = 0$. In physics, this is known as the Edwards–Wilkinson limit.
  - Gu posted a paper with a similar result, but not covering the entire subcritical regime.
  - Dunlap, Gu, Ryzhik and Zeitouni posted a paper containing a new and much shorter proof of the Magnen–Unterberger existence theorem for the solution of the KPZ equation in $d \geq 3$.

Open problems

- What happens at criticality?
- Renormalization group arguments suggest that instead of taking $\lambda = \lambda_c$, if we use $\lambda = \lambda_c +$ correction term, where the correction term is appropriately vanishing as $\varepsilon \to 0$, it may be possible to construct a non-Gaussian scaling limit.
- Getting any non-Gaussian scaling limit for 2D KPZ, either in the above sense, or in the Family–Vicsek scaling, or some other scaling — remains an open problem.
- Show that some natural discrete process has a KPZ scaling limit in $d \geq 2$.