Gauge-string duality in lattice gauge theories

Infosys-ICTS Ramanujan Lectures: Lecture 2

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Recall from Lecture 1: Definition of lattice gauge theory

- Let $d =$ dimension of spacetime, and G be a matrix Lie group.
- The lattice gauge theory with gauge group G on a finite set $\Lambda \subseteq \mathbb{Z}^d$ is defined as follows.
- Suppose that for any two adjacent vertices $x, y \in \Lambda$, we have a group element $U(x,y) \in G$, with $U(y,x) = U(x,y)^{-1}$.
- Let $G(\Lambda)$ denote the set of all such configurations.
- A square bounded by four edges is called a plaquette. Let $P(\Lambda)$ denote the set of all plaquettes in Λ.
- For a plaquette $p \in P(\Lambda)$ with vertices x_1, x_2, x_3, x_4 in anti-clockwise order, and a configuration $U \in G(\Lambda)$, define

$$U_p := U(x_1, x_2)U(x_2, x_3)U(x_3, x_4)U(x_4, x_1).$$

- The Wilson action of U is defined as

$$S_{\mathrm{W}}(U) := \sum_{p \in P(\Lambda)} \mathrm{Re}(\mathrm{Tr}(I - U_p)).$$

Definition of lattice gauge theory contd.

- Let σ_Λ be the product Haar measure on $G(\Lambda)$.
- Given $\beta > 0$, let $\mu_{\Lambda,\beta}$ be the probability measure on $G(\Lambda)$ defined as

$$d\mu_{\Lambda,\beta}(U) := \frac{1}{Z} e^{-\beta S_{\mathrm{W}}(U)} d\sigma_\Lambda(U),$$

  where Z is the normalizing constant.
- This probability measure is called the lattice gauge theory on Λ for the gauge group G, with inverse coupling strength β.
- An infinite volume limit of the theory is a weak limit of the above probability measures as $\Lambda \uparrow \mathbb{Z}^d$.

Recall from Lecture 1: Wilson loops

- Consider a lattice gauge theory on $\mathbb{Z}^d$ with gauge group G.
- Let U be a random configuration of group elements attached to edges, drawn from the probability measure defined by this theory.
- Given a loop γ with directed edges $e_1, \ldots, e_m$, the Wilson loop variable W_γ is defined as

$$W_\gamma := \mathrm{Re}(\mathrm{Tr}(U(e_1)U(e_2)\cdots U(e_m))).$$

- The expected value of W_γ is denoted by $\langle W_\gamma \rangle$. Wilson loop expectations are central objects of interest in lattice gauge theories due to their connection with quark confinement.
- In this talk, we will see how to compute such expectations using a string theory on the lattice $\mathbb{Z}^d$, when β is small and $G = SO(N)$, N large.

The master loop equations

- One approach to computing Wilson loop expectations, using a set of recursive equations that are now known as the Makeenko–Migdal equations or the master loop equations, was introduced by Makeenko and Migdal (1979).
- The equations represent a Wilson loop expectation as a polynomial in several other Wilson loop expectations.
- The equations are approximate, not exact, because Makeenko and Migdal assumed (without proof) that the expectation of a product of Wilson loop variables approximately factorizes as a product of expectations.
- This factorization ansatz is actually an asymptotic property that holds only when the dimension of the gauge group tends to infinity.

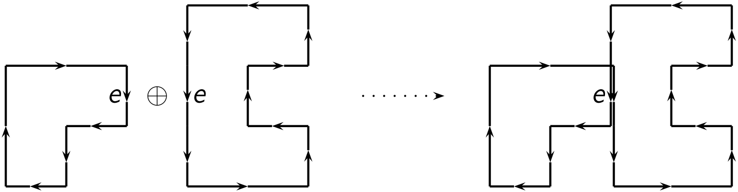
The right approach

- Without the factorization ansatz, a correct set of equations for $U(N)$ lattice gauge theory was derived by Wadia (1981). These equations do not give a closed system, because the terms appearing on the right are not Wilson loop expectations.
- However, if we take our basic objects as *expectations of products of Wilson loop variables instead of expectations of single Wilson loop variables,* then it turns out that one can actually derive a closed system of recursive equations. This idea was used in C. (2015).

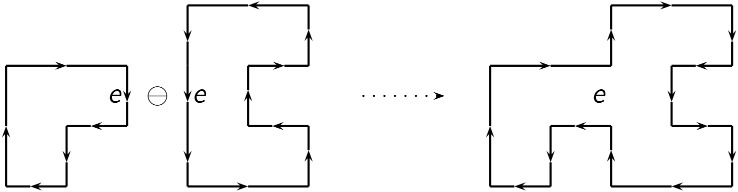
A string theory on the lattice

- The statement of the master loop equations from C. (2015) requires some preparation.
- The first step is the definition of a string theory on $\mathbb{Z}^d$.
- In physics, a string theory prescribes an action for the trajectory of a collection of closed loops (instead of a collection of particles) evolving in time. We will do something analogous on the lattice.
- For us, a string is a finite collection of closed loops in $\mathbb{Z}^d$.
- A string evolves over time.
- At each time step, a component loop may become slightly deformed, or two loops may merge, or a loop may split into two, or a loop may twist at a bottleneck, or nothing may happen.
- Each operation has two subtypes. Total of eight possible operations.
- These operations are described in the following slides.

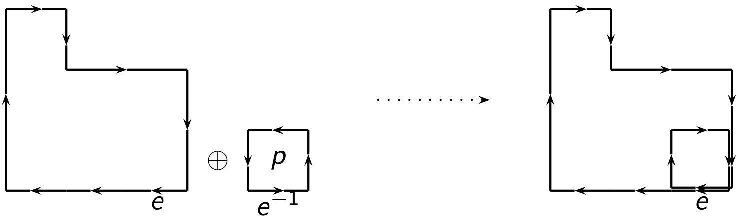
Positive merger



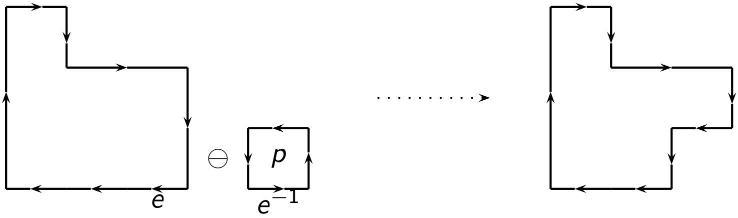
Negative merger



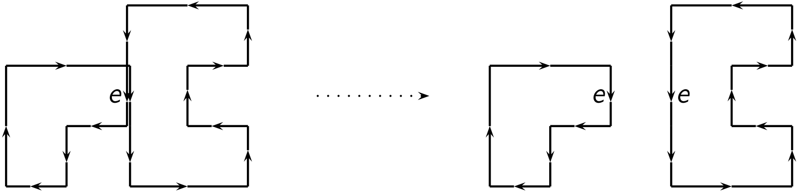
Positive deformation



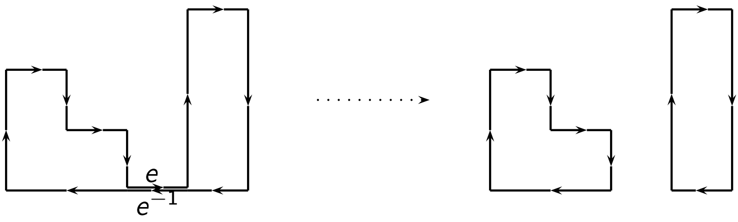
Negative deformation



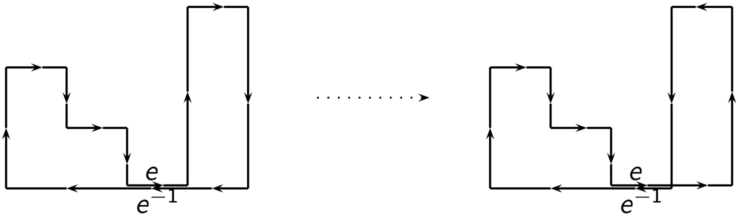
Positive splitting



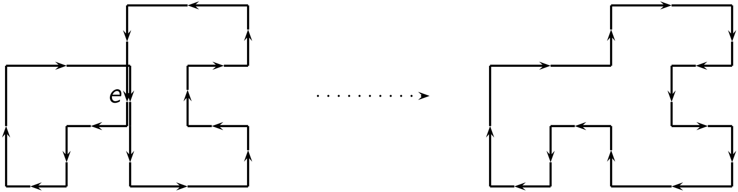
Negative splitting



Positive twisting



Negative twisting



Operations on lattice strings

- After performing one of the eight operations, we get a new string s', which may contain a different number of loops.
- Let

$$
\begin{aligned}
\mathbb{D}^+(s) &:= \{s' : \ s' \text{ is a positive deformation of } s\},\\
\mathbb{D}^-(s) &:= \{s' : \ s' \text{ is a negative deformation of } s\},\\
\mathbb{S}^+(s) &:= \{s' : \ s' \text{ is a positive splitting of } s\},\\
\mathbb{S}^-(s) &:= \{s' : \ s' \text{ is a negative splitting of } s\},\\
\mathbb{M}^+(s) &:= \{s' : \ s' \text{ is a positive merger of } s\},\\
\mathbb{M}^-(s) &:= \{s' : \ s' \text{ is a negative merger of } s\},\\
\mathbb{T}^+(s) &:= \{s' : \ s' \text{ is a positive twisting of } s\},\\
\mathbb{T}^-(s) &:= \{s' : \ s' \text{ is a negative twisting of } s\}.
\end{aligned}
$$

- What is the action in our string theory? We will talk about it later.

The master loop equation

Theorem (C., 2015)

Consider $SO(N)$ lattice gauge theory on $\mathbb{Z}^d$, for any $N \geq 2$ and $d \geq 2$, and any β. For a string $s = (\ell_1, \ell_2, \ldots, \ell_n)$, define

$$\phi(s) := \frac{\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_n} \rangle}{N^n}.$$

Let $|s|$ be the total number of edges in s. Then

$$\begin{aligned}(N-1)|s|\phi(s) &= \sum_{s' \in \mathbb{T}^-(s)} \phi(s') - \sum_{s' \in \mathbb{T}^+(s)} \phi(s') + N \sum_{s' \in \mathbb{S}^-(s)} \phi(s') \\ &- N \sum_{s' \in \mathbb{S}^+(s)} \phi(s') + \frac{1}{N} \sum_{s' \in \mathbb{M}^-(s)} \phi(s') - \frac{1}{N} \sum_{s' \in \mathbb{M}^+(s)} \phi(s') \\ &+ \beta \sum_{s' \in \mathbb{D}^-(s)} \phi(s') - \beta \sum_{s' \in \mathbb{D}^+(s)} \phi(s').\end{aligned}$$

First step in the proof: Integration by parts for $SO(N)$

Lemma (C., 2015)

Let f and g be C^2 functions in an open neighborhood of $SO(N)$ in $\mathbb{R}^{N\times N}$, and let $\mathbb{E}(\cdot)$ denote expectation with respect to the Haar measure. Then

$$\mathbb{E}\biggl(\sum_{i,k} x_{ik}\frac{\partial f}{\partial x_{ik}} g\biggr) = \frac{1}{N-1}\mathbb{E}\biggl(\sum_{i,k}\frac{\partial^2 f}{\partial x_{ik}^2} g - \sum_{i,j,k,k'} x_{jk}x_{ik'}\frac{\partial^2 f}{\partial x_{ik}\partial x_{jk'}} g + \sum_{i,k}\frac{\partial f}{\partial x_{ik}}\frac{\partial g}{\partial x_{ik}} - \sum_{i,j,k,k'} x_{jk}x_{ik'}\frac{\partial f}{\partial x_{ik}}\frac{\partial g}{\partial x_{jk'}}\biggr).$$

Proof of the lemma

- The equation displayed in the previous slide can be written much more compactly as the integration-by-parts identity

$$\int_{SO(N)} (g\Delta f + \langle \nabla g, \nabla f\rangle) d\eta = 0,$$

  where Δ and ∇ are the appropriately defined Laplacian and gradient operators on the manifold $SO(N)$, and η is the normalized Haar measure.
- The previously displayed version is the above equation written in extrinsic coordinates, considering $SO(N)$ as a submanifold of $\mathbb{R}^{N\times N}$.
- This was pointed out to me later by Thierry Lévy. The proof in the paper used a different idea.
- The version in extrinsic coordinates is, however, quite important for subsequent calculations.

Second step in the proof: A property of Wilson loop variables

- Let ℓ be a loop and e be an edge of ℓ.
- Denote the $(i,j)^{\mathrm{th}}$ element of $U(e)$ by q_{ij}.
- Let m is the number of occurrences of e and e^{-1} in ℓ.
- Then, it is not hard to show that

$$mW_\ell = \sum_{i,j} q_{ij} \frac{\partial W_\ell}{\partial q_{ij}}.$$

- This holds because W_ℓ is a homogeneous polynomial of degree m in the q_{ij}'s, if all other matrices are held constant. Whenever $p(x_1, \ldots, x_k)$ is a homogeneous polynomial of degree m in variables $x_1, \ldots, x_k$, we have

$$\sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i} = mp(x_1, \ldots, x_k).$$

Completing the proof

- Let $s = (\ell_1, \ldots, \ell_n)$ be a string.
- Let e be an edge of ℓ_1. Let m be the number of occurrences of e and e^{-1} in ℓ_1. Denote the $(i,j)^{\mathrm{th}}$ element of $U(e)$ by q_{ij}.
- Let g be the product of $W_{\ell_2} W_{\ell_3} \cdots W_{\ell_n}$ with the probability density of $SO(N)$ lattice gauge theory.
- If $\langle \cdot \rangle$ is expectation in the lattice gauge theory and $\mathbb{E}(\cdot)$ is expectation with respect to Haar measure, then by the previous slide,

$$m\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_n} \rangle = m\mathbb{E}(W_{\ell_1} g) = \mathbb{E}\biggl(\sum_{i,j} q_{ij} \frac{\partial W_{\ell_1}}{\partial q_{ij}} g\biggr).$$

- One can now apply the lemma to the right side.
- Somewhat miraculously, the result is the expected value of a polynomial in Wilson loops variables.
- There is nothing special about ℓ_1 and e. The master loop equation is obtained by averaging over all loops and edges.

Remarks

- The eight string operations arise naturally from the above calculation.
- There seems to be no particular reason why the integration by parts should yield an expression involving expected values of products of Wilson loops.
- My former student Jafar Jafarov worked out the master loop equations for $SU(N)$ theory along the same lines. Interestingly, the equations and the string operations are somewhat different for $SU(N)$.
- Open problem: Is there a general result that works for other Lie groups?

The large N limit

- An important direction of research, pioneered by 't Hooft (1974), is to look at very high dimensional gauge groups, such as $SO(N)$ or $SU(N)$ with $N \to \infty$.
- This is known as large N gauge theory.
- The reason for taking $N \to \infty$ is that it simplifies many calculations.
- 't Hooft's work also pioneered the theory of planar diagrams and their connections with random matrices.

Planar diagrams

- Recall the perturbative calculations in quantum field theories, mentioned in the first lecture.
- It was mentioned that the calculations are handled using Feynman diagrams.
- 't Hooft realized that for gauge theories, a slight modification of the Feynman diagram method converts the diagrams into surfaces with edges, vertices and faces.
- Then, 't Hooft suggested that the parameter β should be replaced by $N\beta$. This is known as the 't Hooft scaling.
- This reparametrization has the interesting effect that when N is taken to ∞, the perturbative series expansion becomes a series in powers of $1/N$. This is known as the $1/N$ expansion.
- The interesting outcome of the above exercise is that the k^{th} term in the expansion is a sum over surfaces of genus k. This is the beginning of planar diagram theory.

Planar diagrams in mathematics

- 't Hooft's planar diagram theory has inspired a large body of work in mathematics. Examples include the works of Eynard and Orantin, as well a large body of random matrix literature produced by Guionnet and collaborators.
- However, these results do not deal with gauge theories, which was 't Hooft's original motivation.

Wilson loop expectations in the large N limit

- Recall that a string s is a collection of loops $(\ell_1, \ldots, \ell_n)$. Let

$$\phi_N(s) = \frac{\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_n} \rangle}{N^n}.$$

- Here the expectation is taken with respect to $SO(N)$ lattice gauge theory on $\mathbb{Z}^d$.
- Suppose that like 't Hooft, we replace β by $N\beta$, and send $N \to \infty$.
- Does $\phi_N(s)$ converge to a limit? If so, can we calculate it?
- Observe that since $|\phi_N(s)| \leq 1$, and there are countably many strings, one can always produce a subsequential limit simultaneously for all strings by a diagonal argument.

Master loop equation in the large N limit

- Let $\phi(s)$ be a subsequential limit of $\phi_N(s)$ (simultaneously for all s, under the 't Hooft scaling).
- Then ϕ satisfies the limiting master loop equation

$$|s|\phi(s) = \sum_{s' \in \mathbb{S}^-(s)} \phi(s') - \sum_{s' \in \mathbb{S}^+(s)} \phi(s') + \beta \sum_{s' \in \mathbb{D}^-(s)} \phi(s') - \beta \sum_{s' \in \mathbb{D}^+(s)} \phi(s').$$

- Note that only the splitting and deformation terms have survived, and the twists and mergers have disappeared in the large N limit.
- As before, note that this holds for any β.
- If this system of equations has a unique solution, then we may conclude that $\phi_N(s) \to \phi(s)$ as $N \to \infty$.

Existence of the large N limit at strong coupling

- It was shown in C. (2015) that the system of limiting master loop equations indeed has a unique solution when β is sufficiently small.
- Therefore, $\phi_N(s)$ indeed converges to a limit for every s when β is small.
- Open problem: Show that the limit exists for any β.

Sketch of proof

- Let ϕ and ψ be two solutions of the limiting master loop equations.
- Let Δ be the set of all finite sequences of positive integers.
- If $\delta, \delta' \in \Delta$, we will say that $\delta \le \delta'$ if the two sequences have the same length and δ' dominates δ in each component.
- For a string $s = (\ell_1, \ell_2, \ldots, \ell_n)$, let $\delta(s) = (|\ell_1|, |\ell_2|, \ldots, |\ell_n|)$.
- For $\delta \in \Delta$, let

$$D(\delta) := \sup_{s:\delta(s)\le\delta} |\phi(s) - \psi(s)|.$$

- For $\delta = (\delta_1, \ldots, \delta_n) \in \Delta$, let

$$\iota(\delta) := \delta_1 + \cdots + \delta_n - n.$$

- For $\lambda \in (0, 1)$, let

$$F(\lambda) := \sum_{\delta\in\Delta} \lambda^{\iota(\delta)} D(\delta).$$

Proof sketch contd.

- First, it is shown that $F(\lambda) < \infty$ if λ is sufficiently small.
- Next, repeated applications of the master loop equations produce the inequality

$$F(\lambda) \le \left(4\lambda^3 + 4\lambda + \frac{4\beta d}{\lambda^4} + \frac{4\beta d}{1-\lambda}\right) F(\lambda).$$

- This shows that if λ is small and β is small enough (depending on λ and d), then the coefficient of $F(\lambda)$ on the right is less than 1.
- Due to the finiteness of $F(\lambda)$, this implies that $F(\lambda) = 0$.
- Consequently, $D(\delta) = 0$ for all δ, and hence $\phi = \psi$.

Gauge-string duality in $SO(N)$ lattice gauge theory

- A duality between $SO(N)$ lattice gauge theory with large N, and the lattice string theory described earlier, was established in C. (2015). This may be the first rigorously proved gauge-string duality result. This result will be described in the next few slides.
- In our lattice string theory, call the trajectory of a string vanishing if it vanishes in the finite number of steps.
- Let $X = (s_0, s_1, \ldots, s_k)$ be a vanishing trajectory.
- The action of X is defined as

$$w(X) := w(s_0, s_1)w(s_1, s_2) \cdots w(s_{k-1}, s_k),$$

where $w(s_i, s_{i+1})$ is defined as follows.

Weight of a transition

- Let s be a string and $|s|$ be the total number of edges in s, counting repetitions.
- If s evolves into s' after an operation, define the weight of the transition from s to s' as

$$w(s,s') := \begin{cases} -1/|s| & \text{if } s' \in \mathbb{T}^+(s) \cup \mathbb{S}^+(s) \cup \mathbb{M}^+(s), \\ 1/|s| & \text{if } s' \in \mathbb{T}^-(s) \cup \mathbb{S}^-(s) \cup \mathbb{M}^-(s), \\ -\beta/|s| & \text{if } s' \in \mathbb{D}^+(s), \\ \beta/|s| & \text{if } s' \in \mathbb{D}^-(s). \end{cases}$$

- As mentioned in the previous slide, the action of a vanishing trajectory is computed by multiplying the weights of the successive transitions.

The duality result

Theorem (C., 2015)

Consider $SO(N)$ lattice gauge theory on $\mathbb{Z}^d$ with coupling parameter $N\beta$. Let s be a string and let

$$\phi_N(s) = \frac{\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_n} \rangle}{N^n},$$

as before. Then, if β is small enough, we have

$$\lim_{N\to\infty} \phi_N(s) = \sum_{X \in \mathcal{X}(s)} w(X),$$

where $\mathcal{X}(s)$ is the set of all vanishing trajectories starting at s whose steps consist of only splitting and deformations, and $w(X)$ denotes the action of a trajectory X, as defined earlier. Moreover, the infinite sum on the right is absolutely convergent.

Remarks

- The condition that trajectories can only have deformations and splittings, and no mergers, ensure that the surfaces traced out by the trajectories have genus zero.
- In later work with Jafar Jafarov, we computed the full $1/N$ expansion. The higher order terms involves sums over trajectories that trace out surfaces of high genus.
- Although the formulas look similar to 't Hooft's sums over planar diagrams, they are actually quite different. 't Hooft's analysis and the subsequent body of work look at perturbative expansions in the continuum theory, whereas this result is a non-perturbative theorem for lattice gauge theories.
- Another difference with the planar diagram approach is that it typically gives the result in the form of a divergent series, whereas the theorem presented here comes with a guarantee of convergence.

Proof sketch

- For a string $s = (\ell_1, \ldots, \ell_n)$, define its 'index'

$$\iota(s) := |\ell_1| + \cdots + |\ell_n| - n\,.$$

- We prove facts about strings by induction on the index.
- The key results that help us carry out the inductions is that $\iota(s) > 0$ for any nonempty s, and if s' is a string that is produced by splitting s, then $\iota(s') < \iota(s)$.

Proof sketch, contd.

- The next step is to define a collection of coefficients $a_k(s)$, one for each nonnegative integer k and string s, using a certain inductive definition (by induction on k and $\iota(s)$, as described above) that guarantees the following two properties:
  - For β sufficiently small, the power series

    $$\phi(s) := \sum_{k=0}^{\infty} a_k(s)\beta^k$$

    converges absolutely for any s.
  - The function ϕ satisfies the limiting master loop equations.
- These two properties and the uniqueness of the solution of the master loop equation imply that $\phi_N \to \phi$.

Proof sketch, contd.

- For each $k \geq 0$ and string s, let $\mathcal{X}_k(s)$ be the set of all vanishing trajectories with only deformations and splittings that start at s and have exactly k deformations.
- We show by induction that for any k and s,

$$a_k(s)\beta^k = \sum_{X \in \mathcal{X}_k(s)} w(X).$$

- Once we have this, it follows that

$$\phi(s) = \sum_{k=0}^{\infty} \sum_{X \in \mathcal{X}_k(s)} w(X).$$

- The absolute convergence of the right side requires a different argument that I will omit here.

Open problems

- Develop a continuum version of these results.
- At least, develop a version for large β.
- There are results of Basu and Ganguly (2016) that allow exact computations of the sum over trajectories in 2D using combinatorial connections and free probability theory. Extension to higher dimensions?
- As mentioned before, even establishing the existence of the limit for large β would be a significant achievement.