Mass generation by the Higgs mechanism

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Yang–Mills theories

- Quantum Yang–Mills theories are the building blocks of the Standard Model of quantum mechanics.
- While these are quantum theories in Minkowski space, there are analogues in Euclidean space, known as Euclidean Yang–Mills theories.
- Physicists believe that calculations in the Euclidean theories can be carried over to the quantum theories. (The constructive QFT program aims to justify this mathematically, although it is not yet well-developed for Yang–Mills theories.)
- Euclidean YM theories are supposed to be classical statistical mechanical models of field theories.
- The problem is, we do not even know how to construct these classical models in a rigorous way.

Lattice Yang–Mills theories

- Lattice YM theories, also known as lattice gauge theories, are discretized versions of Euclidean YM theories that are mathematically well-defined.
- In the next few slides, I will define lattice gauge theories.
- Lattice gauge theories model the behaviors of three kinds of fields — gauge fields, boson fields, and fermion fields.
- In this seminar, I will only talk about theories with gauge fields and boson fields.

Gauge field configurations

- Let G be a compact Lie group, called the gauge group of the theory.
- Let Λ be a subset of the lattice $\mathbb{Z}^d$, where d is the dimension of spacetime in the theory. We will take $\Lambda = \{-L, \ldots, L\}^d$ for some L which will be eventually sent to infinity.
- In this talk, we will consider Λ with periodic boundary condition, meaning that we identify opposite faces of Λ.
- Let E denote the set of edges of Λ that are oriented in then positive direction (i.e., $e = (x, y)$ where $x \prec y$ in the lexicographic ordering).
- The space of gauge field configurations of the theory is G^E — that is, a configuration is an assignment of group elements to edges.
- Take any $U = (U_e)_{e \in E} \in G^E$. If $e = (x, y) \in E$ and $e' = (y, x)$, then we define $U_{e'} := U_e^{-1}$.

Plaquettes

- A plaquette in Λ is a square bounded by four edges.
- Given $U \in G^E$ and a plaquette p bounded by four directed edges e_1, e_2, e_3, e_4 joined end-to-end, we define

$$U_p := U_{e_1} U_{e_2} U_{e_3} U_{e_4}.$$
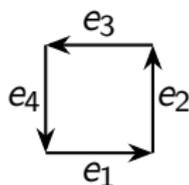
Figure: A plaquette bounded by four directed edges joined end-to-end.

- Note that there is an ambiguity here about the choice of the first edge and the direction of traversal, but that would be immaterial because we will only deal with the quantities that are not affected by these choices.

Higgs field

- Let K be a positive integer. A Higgs field ϕ is a map from Λ into $\mathbb{C}^K$.
- Let σ be a representation of G in $\mathbb{C}^K$, which is unitary with respect to the usual inner product on $\mathbb{C}^K$.
- The representation σ is used to couple the Higgs field to the gauge field (details in the next slide).
- The common parlance is that the Higgs field transforms in the representation σ.
- Let $W : [0, \infty) \to \mathbb{R}$ be a function, called the Higgs potential, that grows faster than quadratic at infinity.
- A configuration in our theory is a pair (U, ϕ), consisting of a gauge field U and a Higgs field ϕ.
- Let $\Sigma := G^E \times (\mathbb{C}^K)^\Lambda$ denote the space of configurations.

Lattice YM theory with a Higgs field

- Fix a parameter $g > 0$, called the gauge coupling constant, and a parameter $\alpha > 0$, called the Higgs length.
- The YM action is a map $S : \Sigma \to \mathbb{R}$ defined as

$$
S(U, \phi) := \frac{1}{2g^2} \sum_{p \in P} \operatorname{Re}(\chi_\rho(U_p)) + \frac{\alpha^2}{2} \sum_{e=(x,y) \in E} \operatorname{Re}(\phi_x^* \sigma(U_e) \phi_y)
$$
$$
- \sum_{x \in \Lambda} W(\|\phi_x\|).
$$

- The theory defines a probability measure μ on Σ as

$$
d\mu(U, \phi) = Z^{-1} e^{S(U,\phi)} \prod_{e \in E} dU_e \prod_{x \in \Lambda} d\phi_x,
$$

where dU_e denotes normalized Haar measure on G, $d\phi_x$ denotes Lebesgue measure on $\mathbb{C}^K$, and Z is the normalizing constant.

Example: $SU(2)$ theory with a Higgs field

- Let $G = SU(2)$, $K = 2$, and $\rho = \sigma =$ the fundamental representation of $SU(2)$.
- Let $W = 0$ on the unit sphere S^3 of $\mathbb{C}^2$ and ∞ outside, so that the Higgs field takes values in S^3.
- The YM action becomes

$$S(U, \phi) := \frac{1}{2g^2} \sum_{p \in P} \operatorname{Re}(\operatorname{Tr}(U_p)) + \frac{\alpha^2}{2} \sum_{e=(x,y) \in E} \operatorname{Re}(\phi_x^* U_e \phi_y).$$

- The theory defines a probability measure μ on $SU(2)^E \times (S^3)^\Lambda$ as

$$d\mu(U, \phi) = Z^{-1} e^{S(U,\phi)} \prod_{e \in E} dU_e \prod_{x \in \Lambda} d\phi_x,$$

where dU_e denotes normalized Haar measure on $SU(2)$, $d\phi_x$ denotes Lebesgue measure on S^3, and Z is the normalizing constant.

Gauge symmetry

- Let $\Theta := G^\Lambda$. That is, an element $\theta \in \Theta$ assigns group elements to vertices. Elements of Θ are called gauge transforms.
- Gauge transforms form a group under pointwise multiplication, and act on Σ as follows.
- Given $\theta \in \Theta$ and $(U, \phi) \in \Sigma$, the configuration $(V, \psi) = \theta(U, \phi)$ is defined as

$$V_e := \theta_x U_e \theta_y^{-1}, \quad \psi_x := \sigma(\theta_x)\phi_x,$$

where e is the edge (x, y).

- It turns out that any gauge transform θ preserves the measure μ defined by our lattice YM theory. This property of lattice YM theories is known as gauge symmetry.
- Gauge symmetry is considered to be an essential feature of any gauge theory. A model that does not posses gauge symmetry is unphysical.
- Moreover, any physically relevant observable must be gauge invariant, i.e., invariant under gauge transforms.

Gauge fixing

- Two configurations are called gauge equivalent if one can be obtained from the other by the action of a gauge transformation.
- Gauge fixing is any way of choosing one element from each gauge equivalence class.

Gauge fixing in $SU(2)$ Yang–Mills-Higgs theory

- Given a configuration (U, ϕ), we choose an element from its gauge equivalence class as follows.
- For each $x \in \Lambda$, let θ_x be the unique element of $SU(2)$ such that

$$\theta_x \phi_x = e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

  (It is easy to show that θ_x is unique.)
- Let θ be the gauge transform $(\theta_x)_{x \in \Lambda}$.
- Let $(V, \psi) := \theta(U, \phi)$.
- It is easy to check that $\psi_x = e_1$ for all x.
- Thus, if our interest lies only in gauge invariant observables, it suffices to study the gauge field V.
- The above procedure is known as fixing the unitary gauge.

Mass gap

- A statistical mechanical theory is said to have a mass gap (or simply, gapped) if it has exponential decay of correlations.
- The inverse of the decay exponent is called the correlation length of the theory.
- To obtain a continuum limit of a theory on the lattice, we need to take a scaling limit (i.e., vary the parameters of the models while sending the lattice spacing to zero) such that the correlation length tends to infinity (and the lattice spacing is scaled as the inverse of the correlation length).
- Usually, a physically relevant theory is obtained only upon taking a continuum limit.

Mass gap at strong coupling

- Using the cluster expansion technique of Glimm, Jaffe and Spencer (1976), it was shown by Osterwalder and Seiler (1978) that any pure lattice YM theory (i.e., a theory without a Higgs field or fermions) has a mass gap when g is large enough.

- This is known as the strong coupling regime.

- However, to have the correlation length tend to infinity, we need to take $g \to 0$, also known as the weak coupling limit.

- Pure lattice YM theories need not be gapped at weak coupling.

- For example, 4D pure $U(1)$ theory is the theory of photons. It is massive at strong coupling, but has no mass at weak coupling — which is what should be the case, because photons have no mass.

- The Higgs field was introduced as a way of generating mass in the continuum limit, while preserving gauge symmetry.

Known results about the Higgs mechanism

- The Higgs mechanism in $d = 2$ is well-understood rigorously when the gauge group is $U(1)$, due to the works of many authors in the late 70s and early 80s (Balaban, Borgs, Brydges, Fröhlich, Imbrie, Jaffe, Seiler, ...).

- For non-Abelian theories satisfying a condition called complete breakdown of symmetry, Seiler (1982) proved mass gap in arbitrary dimension, provided that αg is large enough.

- Unfortunately, holding αg larger than a constant prevents the correlation length from diverging, even if we take $g \to 0$.

- Thus, this does not suffice for proving mass generation by the Higgs mechanism in a continuum limit.

Towards the main result

- The main result that I am going to present is that for $SU(2)$ Yang–Mills–Higgs theory, one can take $\alpha \to \infty$ and $g \to 0$ in such a way that the correlation length diverges, and upon taking the scaling limit of the field V (obtained by unitary gauge fixing), we obtain a continuum limit object which has a mass gap. The result is valid in any $d \geq 2$.

- This proves the validity of the Higgs mechanism in a continuum limit (and is the first such proof in $d \geq 3$).

- The scaling limit, however, is Gaussian, which means that it does not allow particle interactions in a quantum mechanical sense.

- This is reminiscent of constructions by Gross (1983) and Driver (1987) of Gaussian scaling limits of $U(1)$ theory (without a Higgs field) in $d = 3$ and $d = 4$.

- It is not clear if a non-Gaussian limit is at all possible in $d = 4$ (especially after the Aizenman–Duminil-Copin proof of the triviality of ϕ_4^4). But at least in $d = 3$, it may be possible.

Stereographic projection

- Consider the following variant of stereographic projection of the unit sphere $S^3 \subseteq \mathbb{R}^4$.
- For each $x \in S^3 \setminus \{-e_1\}$ (where $e_1 = (1, 0, 0, 0)$), take the line passing through x and $-e_1$, and let $y = (y_1, y_2, y_3, y_4)$ be the unique point at the intersection of this line and the plane $x_1 = 1$.
- Let $\sigma_3(x) := (y_2, y_3, y_4) \in \mathbb{R}^3$.
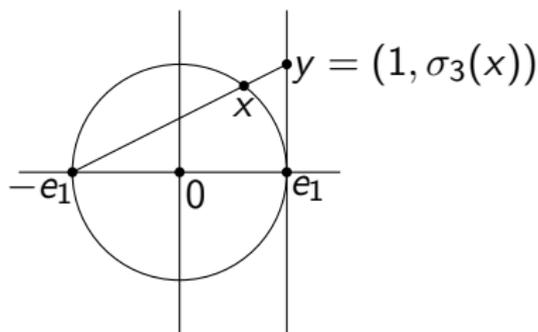- Note that σ_3 defines a bijection between $S^3 \setminus \{-e_1\}$ and $\mathbb{R}^3$.

Figure: Stereographic projection $\sigma_3 : S^3 \to \mathbb{R}^3$.

- Recall that any element $U \in SU(2)$ can be uniquely written as

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

  where $a, b \in \mathbb{C}^2$ and $|a|^2 + |b|^2 = 1$.

- Writing $a = x + iy$ and $b = w + iz$, we see that this gives a one-to-one correspondence between $SU(2)$ and S^3, which we call τ.

- That is, $\tau(U) = (x, y, w, z)$.

- We can further compose this with the stereographic projection described in the previous slide, to obtain a one-to-one correspondence $\sigma_3 \circ \tau$ between $SU(2) \setminus \{-I\}$ and $\mathbb{R}^3$.

Transforming the gauge field of $SU(2)$ theory

- Recall the gauge field V in our $SU(2)$ theory after unitary gauge fixing.
- For each $e \in E$, let

$$A_e := \frac{\sqrt{2}}{g} \sigma_3(\tau(V_e)).$$

- Note that $A_e = (A_e^1, A_e^2, A_e^3)$ is an element of $\mathbb{R}^3$.
- For each $x \in \mathbb{Z}^d$, $1 \leq i \leq d$, and $1 \leq j \leq 3$, let

$$Y_i^j(x) := A_e^j,$$

  where e is the edge $(x, x + e_i)$ (where $e_1, \ldots, e_d$ denote the standard basis vectors of $\mathbb{R}^d$), assuming $e \in E$. If $e \notin E$, let $Y_i^j(x) := 0$.
- For $j = 1, 2, 3$, let Y^j denote the field $(Y_i^j(x))_{x \in \mathbb{Z}^d, 1 \leq i \leq d}$.
- Let $Y := (Y^1, Y^2, Y^3)$.
- For $x \in \mathbb{R}^d \setminus \mathbb{Z}^d$, define $Y(x) := Y(x')$, where x' is the point in $\mathbb{Z}^d$ that is closest to x (ignoring the set of measure zero where x' is not unique).

Main result

Theorem (C., 2024)

Take any $d \geq 2$. Take $\alpha = g^{\kappa - 1}$ for some $\kappa \in (0, 1/49d)$. Suppose that $g \to 0$ and $L \to \infty$ in such a way that $Lg^{3\kappa} \to \infty$. Let $\varepsilon := c\alpha g$ for some fixed $c > 0$. Then the field $Z(x) := \varepsilon^{-(d-2)/2} Y(\varepsilon^{-1} x)$ converges in law to a triple of independent Euclidean Proca fields with mass $1/(\sqrt{2}c)$ (to be defined in the subsequent slides).

Euclidean Proca field

- The Euclidean Proca field is a variant of the Proca field appearing in quantum field theory, obtained by replacing Minkowski spacetime by Euclidean spacetime.
- The Euclidean Proca field was defined by L. Gross (1974) and Ginibre and Velo (1975).
- One way to define it is as follows.
- Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions on $\mathbb{R}^d$.
- Let $\mathcal{A}(\mathbb{R}^d) := (\mathcal{S}(\mathbb{R}^d))^d$ be the space of Schwartzian 1-forms.
- Let us denote an element $f = (f_1, \ldots, f_d) \in \mathcal{A}(\mathbb{R}^d)$ as

$$f = f_1 dx_1 + \cdots + f_d dx_d,$$

following the usual convention of denoting 1-forms.

- The Euclidean Proca field on $\mathbb{R}^d$ with parameter $\lambda > 0$, for $d \geq 3$, is a random linear functional X on $\mathcal{A}(\mathbb{R}^d)$ having the property that for any $f \in \mathcal{A}(\mathbb{R}^d)$, $X(f)$ is a Gaussian random variable with mean zero and variance

$$\sum_{i=1}^{d} \int f_i(x)(-\Delta + \lambda I)^{-1} f_i(x) dx$$

$$+ \frac{1}{\lambda} \sum_{i=1}^{d} \sum_{j=1}^{d} \int \partial_i f_i(x)(-\Delta + \lambda I)^{-1} \partial_j f_j(x) dx.$$

Mass of the Euclidean Proca field

- For any $f \in \mathcal{A}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, let f^x denote the translation of f by x. That is, $f^x(y) := f(x + y)$.

Theorem (C., 2024)

Let X be a Euclidean Proca field on $\mathbb{R}^d$ with parameter λ. Let $\{x_n\}_{n \geq 1}$ be a sequence in $\mathbb{R}^d$ such that $\|x_n\| \to \infty$ and $\|x_n\|^{-1} x_n \to u \in S^{d-1}$. Then for any compactly supported $f, g \in \mathcal{A}(\mathbb{R}^d)$,

$$\lim_{n \to \infty} \frac{\mathbb{E}(X(f) X(g^{x_n}))}{\|x_n\|^{-(d-1)/2} e^{-\sqrt{\lambda} \|x_n\|}} = \Psi(f, g, u, \lambda),$$

where $\Psi(f, g, u, \lambda)$ is an explicit function of f, g, u and λ.

- In particular, the field has mass $\sqrt{\lambda}$.

Proof idea for the main result

- Let $\Sigma := SU(2)^E$.
- Let Σ' be the subset of Σ consisting of all U such that $\|U_e - I\| \leq \alpha^{-1}$ for all e.
- The normalized Haar measure of Σ' is $\geq (C_1\alpha)^{-C_2 L^d}$.
- One can show that the probability density of V is proportional to

$$p(U) := \exp\left(-\frac{1}{2g^2} \sum_{p \in P} \|U_p - I\|^2 - \frac{\alpha^2}{4} \sum_{e \in E} \|U_e - I\|^2 \right).$$

- If $U \in \Sigma'$, then $p(U) \geq e^{-C_3 L^d/\alpha^2 g^2}$.
- Combining, we get

$$\int p(U) \prod_{e \in E} dU_e \geq e^{-C_3 L^d/\alpha^2 g^2} (C_1\alpha)^{-C_2 L^d}.$$

Proof idea, contd.

- Consequently,

$$\frac{1}{\int p(U) \prod_{e \in E} dU_e} \leq e^{C_3 L^d / \alpha^2 g^2} (C_1 \alpha)^{C_2 L^d}.$$

- But

$$\frac{1}{\int p(U) \prod_{e \in E} dU_e} = \frac{\int p(U)^{-1} p(U) \prod_{e \in E} dU_e}{\int p(U) \prod_{e \in E} dU_e} = \mathbb{E}(p(V)^{-1}).$$

- Let $H(U) := -\log p(U)$.

- Then for any $t \geq 0$,

$$\mathbb{P}(H(V) \geq t) = \mathbb{P}(p(V)^{-1} \geq e^t) \leq e^{-t} \mathbb{E}(p(V)^{-1})$$
$$\leq e^{-t} e^{C_3 L^d / \alpha^2 g^2} (C_1 \alpha)^{C_2 L^d}.$$

Proof idea, contd.

- But also, $\mathbb{P}(H(V) \geq t) \leq 1$.
- Thus,

$$
\begin{aligned}
\mathbb{E}(H(V)) &= \int_0^\infty \mathbb{P}(H(V) \geq t) dt \\
&\leq \int_0^\infty \min\{e^{-t} e^{C_3 L^d / \alpha^2 g^2} (C_1 \alpha)^{C_2 L^d}., 1\} dt \\
&\leq \frac{C_4 L^d}{\alpha^2 g^2} + C_5 L^d \log \alpha.
\end{aligned}
$$

- But $H(V) \geq \frac{\alpha^2}{4} \sum_{e \in E} \|V_e - I\|^2$, and by symmetry (due to periodic boundary), $\mathbb{E}\|V_e - I\|^2$ is the same for all e.
- Thus, for any e, we obtain the key estimate

$$
\mathbb{E}\|V_e - I\|^2 \leq \frac{C_6}{\alpha^4 g^2} + \frac{C_7 \log \alpha}{\alpha^2}.
$$

- Note that this bound has no dependence on L.

Proof idea, contd.

- Let $\Lambda' := \{-M, \ldots, M\}^d$, where $M \ll L$.
- Let E' denote the set of edges of Λ'. Then

$$\mathbb{P}(\max_{e \in E'} \|V_e - I\|^2 > \delta) \leq \frac{1}{\delta^2} \sum_{e \in E'} \mathbb{E}\|U_e - I\|^2$$

$$\leq \frac{C_8 M^d}{\alpha^4 g^2 \delta^2} + \frac{C_9 M^d \log \alpha}{\alpha^2 \delta^2}.$$

- Choosing $\alpha = g^{\kappa-1}$, $M = g^{-4\kappa}$ and $\delta = g^{1-6d\kappa}$ for some sufficiently small κ, this bound can be taken to zero as $g \to 0$.
- Armed with this and several other estimates of a similar nature, the rest of the analysis proceeds by perturbative expansion around a Gaussian measure, which scales to the Euclidean Proca field.