Gauge-string duality in lattice gauge theories

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Yang–Mills theories

- Maxwell's equations are a set of four equations that describe the behavior of an electromagnetic field.
- Hermann Weyl showed that these four equations are actually the Euler–Lagrange equations for an elegant minimization problem.
- In modern parlance, Maxwell's equations minimize the Yang–Mills functional for the gauge group $U(1)$.
- Physicists later realized that two of the other three fundamental forces of nature — the weak force and the strong force — can also be modeled by similar equations: one needs to simply change the group $U(1)$ to some other group. ($SU(3)$ for the strong force, $SU(2) \times U(1)$ for the weak force.)
- Equations obtained by minimizing Yang–Mills functionals over the space of connections of a principal bundle.
- Many years of trial and error led to the formulation of the Standard Model of quantum mechanics.

Quantum Yang–Mills theories

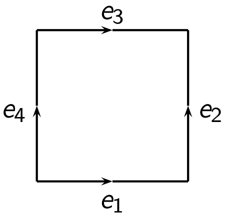
- Quantum Yang–Mills theories are the building blocks of the Standard Model.
- Objective is to define a probability measure with density proportional to $\exp(-\text{Yang–Mills functional})$. This is one of the Clay millennium problems.
- In 1974, Kenneth Wilson introduced discrete versions of quantum Yang–Mills theories. These are known as lattice gauge theories. Rigorously defined objects.
- The dominant approach to giving a rigorous mathematical construction of quantum Yang–Mills theories is by attempting to take a continuum limit of lattice gauge theories.
- Partial progress by many authors in the 80's and 90's: Guth, Fröhlich, Spencer, Brydges, Seiler, Gross, Driver, Sengupta, King, Balaban, Magnen, Rivasseau, Sénéor, and others. Recent progress in two dimensions by Thierry Lévy.

Lattice gauge theories

- A lattice gauge theory involves a lattice, say $\mathbb{Z}^d$, and a compact Lie group G that is called the gauge group for the theory.
- Take a large but finite box Λ in the lattice. Let E be the set of positively oriented edges in Λ.
- Consider the set G^E, the E-fold power of G. Let μ be the product Haar measure on this set.
- A lattice gauge theory introduces a new measure ν on this set, which has a prescribed density f with respect to μ.
- To define f, the theory needs one more component: a matrix representation ρ of G.
- We also need to define the notion of a plaquette.

Plaquettes

- A plaquette p is simply a square in $\mathbb{Z}^d$, surrounded by four positively oriented edges e_1, e_2, e_3, e_4.



- Note that even if $d > 2$, a plaquette is always 2-dimensional.
- Given an element $g = (g_e)_{e \in E} \in G^E$ and a plaquette p as above, define

$$g_p := g_{e_1} g_{e_2} g_{e_3}^{-1} g_{e_4}^{-1} .$$

Definition of lattice gauge theory

- Recall: We have a box Λ in $\mathbb{Z}^d$, E is the set of positively oriented edges in Λ, G is a compact Lie group, μ is the product Haar measure on G^E, and ρ is a matrix representation of G.
- Let P be the set of plaquettes in Λ. Given $g \in G^E$ and $p \in P$, recall definition of g_p.
- Define $f : G^E \to [0, \infty)$ as

$$f(g) := \exp\left(\lambda \sum_{p \in P} \Re \operatorname{Tr} \rho(g_p)\right),$$

  where λ is a parameter, called the coupling constant.
- Let ν be the probability measure on G^E that has density proportional to f with respect to μ.
- The probability measure ν defines a lattice gauge theory.
- For a function $h : G^E \to \mathbb{R}$, define $\langle h \rangle := \int h d\nu$.

Wilson loops

- The main objects of study in lattice gauge theories are Wilson loop variables.
- A Wilson loop ℓ is simply a closed path in the lattice $\mathbb{Z}^d$. For now, we will assume that ℓ lies entirely in the box Λ.
- Recall that an element $g \in G^E$ is an assignment of group elements to positively oriented edges of Λ.
- If e^{-1} is the negatively oriented version of an edge $e \in E$, define $g_{e^{-1}} := g_e^{-1}$.
- One can look at ℓ as a sequence of edges $e_1, e_2, \ldots, e_n$, some of which are positively oriented and others negatively.
- The Wilson loop variable W_ℓ is defined as

$$W_\ell := \Re \operatorname{Tr} \rho(g_{e_1} g_{e_2} \cdots g_{e_n}) .$$

- Key question: Compute $\langle W_\ell \rangle$, where $\langle \cdot \rangle$ denotes expectation with respect to the lattice gauge measure ν defined before.

Quark confinement and Wilson's area law

- Wilson's original motivation was to understand why there are no free quarks in nature. (Quark confinement.)
- By drawing analogy with the continuum theory for the strong force, Wilson argued that if $\langle W_\ell \rangle$ behaves like $\exp(-C \cdot \text{area}(\ell))$ in 4-dimensional $SU(3)$ lattice gauge theory, where $\text{area}(\ell)$ denotes the minimum surface area enclosed by ℓ, then quarks are confined.
- Proving upper bound is actually sufficient.
- Has been rigorously proved by Osterwalder and Seiler (1978) in the case where λ is sufficiently small and ℓ is a rectangle.

Large N gauge theories: 't Hooft's approach

- Gauge groups such as $SU(5)$, $SU(3)$ and $SU(2) \times U(1)$ are the ones that are relevant for physical theories.
- However, theoretical understanding is difficult to achieve.
- Gerardus 't Hooft (1974) suggested a simplification of the problem by considering groups such as $SU(N)$ where N is large.
- The $N \to \infty$ limit simplifies many theoretical problems.
- Calculation of normalizing constants by 't Hooft's planar diagram theory.
- Some aspects of the planar diagram theory for systems with a fixed number of vertices has been made rigorous by various authors (Guionnet, Ercolani, McLaughlin, Eynard, Jones,...).
- Closely related to topological recursion.

Gauge-string duality

- 't Hooft's approach pointed to a connection between gauge theories and string theories.
- In string theories, particles are replaced by one-dimensional strings.
- A string moving through time traces out a surface. From that perspective, string theories may be viewed as theories of random surfaces.
- The planar diagrams in 't Hooft's approach give a relation between computations in gauge theories and counting certain kinds of surfaces.

Gauge-string duality, contd.

- This connection was not very well understood in the physics community until the seminal work of Maldacena (1997), which gave birth to the flourishing field of AdS/CFT duality, sometimes called gauge-gravity duality.
- This theory gives, among other things, an explicit way to write the expectations of Wilson loop variables in a certain supersymmetric gauge theory (the 'CFT' part) in the large N limit in terms of weighted sums over trajectories of strings in a high dimensional string theory (the 'AdS' part).
- None of this is rigorous math, however. Mainly because the underlying objects are not mathematically defined.

Goal of this talk

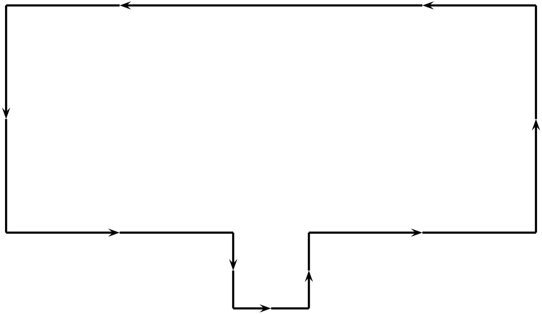
- In the large N limit of $SO(N)$ lattice gauge theory on $\mathbb{Z}^d$, at small enough λ, show that expectations of Wilson loop variables can indeed be written explicitly as sums over trajectories in a certain string theory on the lattice.
- This solves two problems:
  - Theoretical computation of Wilson loop expectations. I have not seen this worked out in a rigorous math setting in dimensions higher than two. (In two dimensions, there is a significant body of work due to Thierry Lévy.)
  - Hint at a general mathematical reason behind gauge-string duality, going beyond exactly solvable models. (An exactly solvable case, for example, is three dimensional Chern–Simons gauge theory analyzed by Witten, preceding Maldacena's work.)

A string theory on the lattice

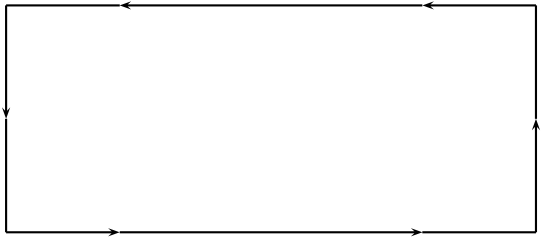
- Basic objects: Collections of finitely many closed paths in $\mathbb{Z}^d$, called 'loop sequences'. Analogous to collections of closed strings in the continuum.
- Loop sequences can evolve in time according to certain rules.
- At each time step, only one loop in a loop sequence is allowed to be modified.
- Four possible modifications:
  - Positive deformation. (Addition of a plaquette without erasing edges.)
  - Negative deformation. (Addition or deletion of a plaquette that involves erasing at least one edge.)
  - Positive splitting. (Splitting a loop into two loops without erasing edges.)
  - Negative splitting. (Splitting a loop into two loops in a way that erases at least one edge.)

Example of a negative deformation

Example of a negative deformation



Another example of a negative deformation



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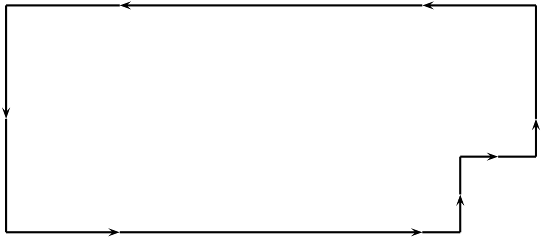
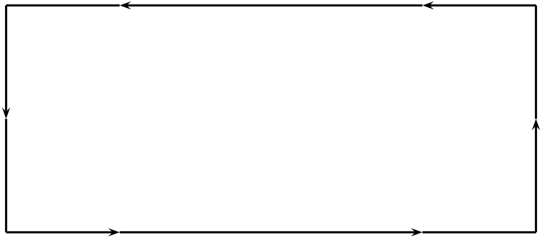


Example of a positive deformation



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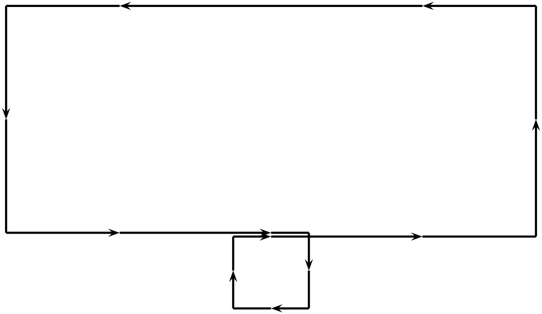
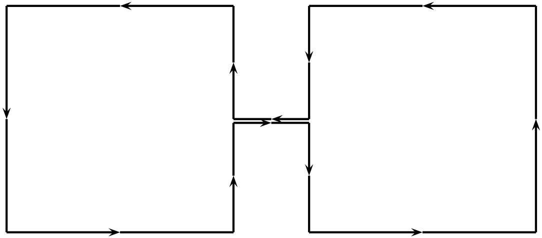


Example of a negative splitting



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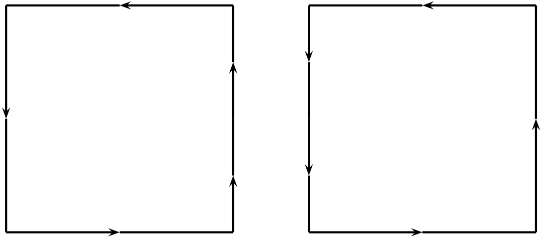

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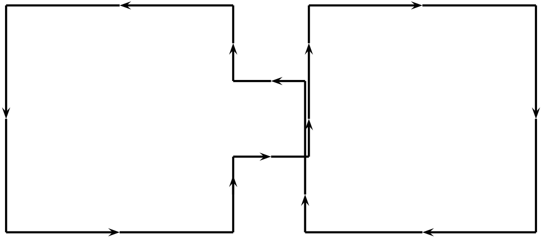

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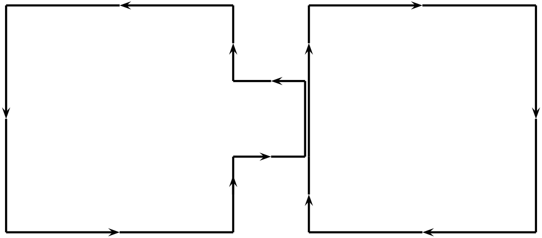
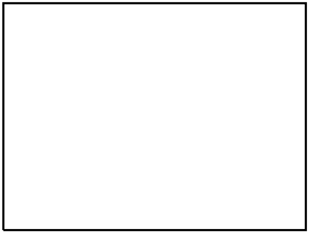


Trajectories

- The evolution of a loop sequence will be called a trajectory.
- A trajectory will be called vanishing if it ends in nothing in a finite number of steps.

Example of a vanishing trajectory starting at a 4×3 rectangle



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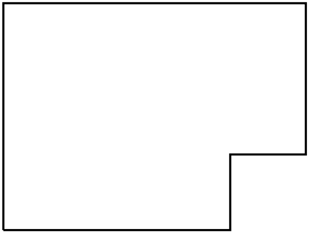

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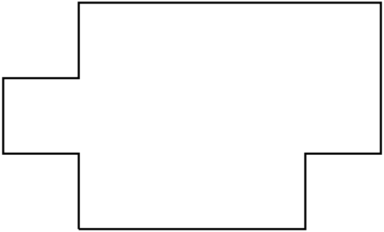

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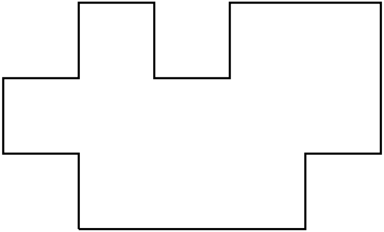

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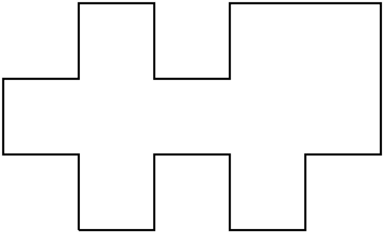

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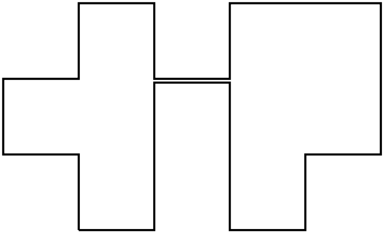

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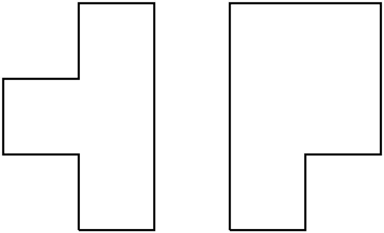

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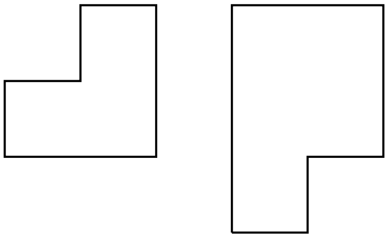

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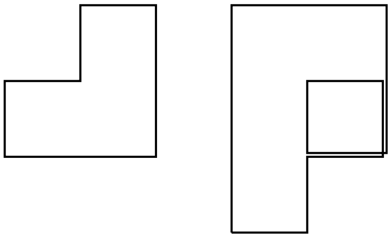

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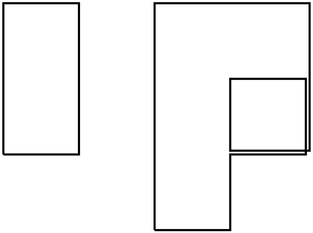

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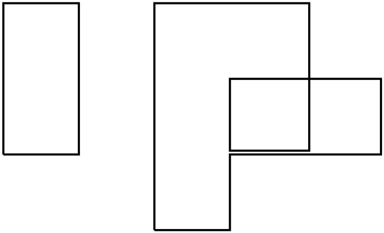

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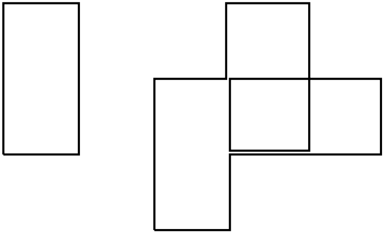

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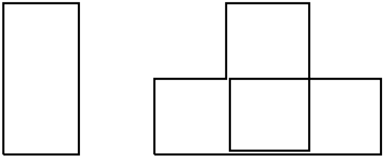

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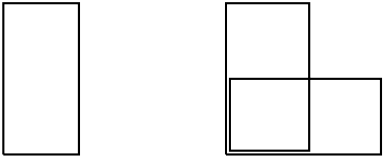

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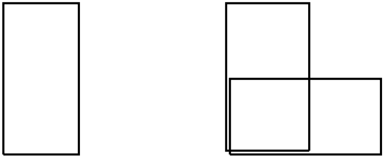

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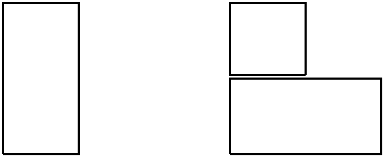

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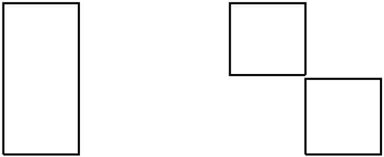

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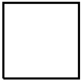

Example of a vanishing trajectory starting at a 4×3 rectangle

Action of a trajectory

- String theories in the continuum prescribe actions for trajectories of strings, analogous to the Lagrangian action of classical mechanics.
- For example, in Bosonic string theory, the Nambu–Goto action is the area of the surface traced out by the string as it moves through time.
- We would like to define an action for a trajectory of loop sequences in our lattice string theory.

Action of a trajectory, contd.

- The lattice string theory defined here has a parameter β.
- Depending on the value of β, each step of a trajectory is given a weight, as follows.
- Let m be the total number of edges in the loop sequence before the step is taken.
- The weight of the step is defined to be:

$$\begin{aligned} -\beta/m &\text{ if the step is a positive deformation;} \\ \beta/m &\text{ if the step is a negative deformation;} \\ -2/m &\text{ if the step is a positive splitting;} \\ 2/m &\text{ if the step is a negative splitting.} \end{aligned}$$

- The weight or action of a vanishing trajectory X is defined to be the product of the weights of the steps in the trajectory. Denoted by $w_\beta(X)$.

Main result

Theorem (C., 2015)

There exists $\beta_0 > 0$ *such that the following is true. Let* ℓ *be a fixed loop in* $\mathbb{Z}^d$. *Let* $\langle W_\ell \rangle$ *denote the expectation of the Wilson loop variable* W_ℓ *in the* $SO(N)$ *lattice gauge theory in a box* $\Lambda_N \subseteq \mathbb{Z}^d$, *with coupling constant* $\lambda = N\beta$. *If* $|\beta| \le \beta_0$ *and* $\Lambda_N \uparrow \mathbb{Z}^d$ *as* $N \to \infty$, *then*

$$\lim_{N\to\infty} \frac{\langle W_\ell \rangle}{N} = \sum_{X \in \mathcal{X}(\ell)} w_\beta(X)\,,$$

where $\mathcal{X}(\ell)$ *is the set of all vanishing trajectories starting at* ℓ *and* $w_\beta(X)$ *is the action of* X *defined earlier. Moreover, the infinite sum on the right is absolutely convergent.*

Implication for Wilson area law

- Recall that area(ℓ) was defined to be the minimum surface area enclosed by a loop ℓ.
- In the context of the lattice, one can take a 'surface' to mean a 2-chain in the standard cell complex of $\mathbb{Z}^d$.
- Then the notion of a surface having boundary ℓ is well-defined, if we look at ℓ as a 1-cycle.
- Since any 2-chain can be written as a linear combination of plaquettes with integer coefficients, the sum of the absolute values of these coefficients give a natural notion of the area of a lattice surface.

Implication for Wilson area law, contd.

- Key observation, making use of the Poincaré lemma: The minimum number of steps required to gradually modify a loop (according the our deformations and splittings) into nothing, is equal to the minimum surface area enclosed by the loop.
- From this and the main theorem, it is not hard to deduce that

$$\lim_{N\to\infty} \frac{\langle W_\ell \rangle}{N} \le e^{-C\cdot \mathrm{area}(\ell)}\,.$$

- This is the first rigorous result that proves the area law upper bound for an arbitrary loop. All previous results are for rectangles.
- A corresponding lower bound should also hold, but seems to be harder to prove because the actions of trajectories may be positive or negative, with potential cancelations.

Neighborhood of a loop sequence

- In the next few slides, I will try to give an outline of the proof of the main theorem.
- In addition to deformations and splittings, I define two additional operations on loop sequences: stitchings and twistings. Will skip the definitions here.
- Given a loop sequence s, define

$$\begin{aligned}
\mathfrak{D}^+(s) &:= \{s' : \ s' \text{ is a positive deformation of } s\}\,,\\
\mathfrak{D}^-(s) &:= \{s' : \ s' \text{ is a negative deformation of } s\}\,,\\
\mathfrak{S}^+(s) &:= \{s' : \ s' \text{ is a positive splitting of } s\}\,,\\
\mathfrak{S}^-(s) &:= \{s' : \ s' \text{ is a negative splitting of } s\}\,,\\
\mathfrak{N}^+(s) &:= \{s' : \ s' \text{ is a positive stitching of } s\}\,,\\
\mathfrak{N}^-(s) &:= \{s' : \ s' \text{ is a negative stitching of } s\}\,,\\
\mathfrak{T}^+(s) &:= \{s' : \ s' \text{ is a positive twisting of } s\}\,,\\
\mathfrak{T}^-(s) &:= \{s' : \ s' \text{ is a negative twisting of } s\}\,.
\end{aligned}$$

The master loop equation

The following is a generalization of what are called Makeenko–Migdal equations or master loop equations in the physics literature on lattice gauge theories.

Theorem (C., 2015)

For a loop sequence $s = (\ell_1, \ldots, \ell_n)$, *define*

$$\phi(s) := \frac{\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_n} \rangle}{N^n}\,.$$

Let $|s|$ *be the total number of edges in* s*. Then*

$$\begin{aligned}(N-1)|s|\phi(s) &= 2\sum_{s' \in \mathfrak{T}^-(s)} \phi(s') - 2\sum_{s' \in \mathfrak{T}^+(s)} \phi(s') + 2N \sum_{s' \in \mathfrak{S}^-(s)} \phi(s') \\ &\quad - 2N \sum_{s' \in \mathfrak{S}^+(s)} \phi(s') + \frac{1}{N}\sum_{s' \in \mathfrak{N}^-(s)} \phi(s') - \frac{1}{N}\sum_{s' \in \mathfrak{N}^+(s)} \phi(s') \\ &\quad + N\beta \sum_{s' \in \mathfrak{D}^-(s)} \phi(s') - N\beta \sum_{s' \in \mathfrak{D}^+(s)} \phi(s')\,.\end{aligned}$$

The main idea

- The loop equation relates the expectation of the Wilson variable for one loop sequence with the expectations of a set of 'neighboring sequences'.
- It does not suffice to work with loops — the recursion is defined on loop sequences.
- This is a key difference with physics papers on Makeenko–Migdal equations, which make a non-rigorous factorization ansatz to circumvent the problem.
- The recursion naturally leads to a formal expression in terms of a sum over trajectories of loop sequences.
- Main challenge is to prove convergence. Involves Catalan numbers and combinatorial arguments. This takes up a substantial part of the paper.
- I will now outline how the equations are obtained via Stein's method.

Stein equation for $SO(N)$

- In the probability literature, the Stein identity for a standard Gaussian variable Z says that $\mathbb{E}(Zf(Z)) = \mathbb{E}(f'(Z))$ for all absolutely continuous f for which the right side is finite. (Known by other names in other fields.)
- The following theorem generalizes this to $SO(N)$.

Theorem (C., 2015)

Let f and g be C^2 functions on $SO(N)$, and let $\mathbb{E}(\cdot)$ denote expectation with respect to the Haar measure. Then

$$\mathbb{E}\biggl(\sum_{i,k} x_{ik}\frac{\partial f}{\partial x_{ik}} g\biggr) = \frac{1}{N-1}\mathbb{E}\biggl(\sum_{i,k}\frac{\partial^2 f}{\partial x_{ik}^2} g - \sum_{i,j,k,k'} x_{jk}x_{ik'}\frac{\partial^2 f}{\partial x_{ik}\partial x_{jk'}} g + \sum_{i,k}\frac{\partial f}{\partial x_{ik}}\frac{\partial g}{\partial x_{ik}} - \sum_{i,j,k,k'} x_{jk}x_{ik'}\frac{\partial f}{\partial x_{ik}}\frac{\partial g}{\partial x_{jk'}}\biggr).$$

How to prove the loop equation

- Fix some edge $e \in \ell$.
- Let $Q = (q_{ij})_{1 \le i,j \le N}$ be the element of $SO(N)$ attached to e.
- Fact: If m is the number of occurrences of e and e^{-1} in ℓ, then
$$W_\ell = m \sum_{i,j} q_{ij} \frac{\partial W_\ell}{\partial q_{ij}}\,.$$
- Let g be the density of the lattice gauge measure with respect to the product Haar measure on $SO(N)^{E(\Lambda_N)}$.
- If $\langle \cdot \rangle$ is expectation in the lattice gauge theory and $\mathbb{E}(\cdot)$ is expectation with respect to the product Haar measure, then
$$\langle W_\ell \rangle = \mathbb{E}(W_\ell g) = m\,\mathbb{E}\biggl(\sum_{i,j} q_{ij} \frac{\partial W_\ell}{\partial q_{ij}} g\biggr)\,.$$
- One can now apply the Stein equation to the right-hand side. It turns out that the resulting identity is exactly the master loop equation that was written down earlier.

Ongoing work

- Work in progress with Jafar Jafarov: Make a rigorous version of 't Hooft's $1/N$ expansion in the strong coupling regime.
- That is, find $f_0, f_1, \ldots$ such that

$$\frac{\langle W_\ell \rangle}{N} = f_0(\ell) + \frac{f_1(\ell)}{N} + \cdots + \frac{f_k(\ell)}{N^k} + O\left(\frac{1}{N^{k+1}}\right).$$

- f_0 has already been identified in this talk as a sum over trajectories. These trajectories allows loops to only deform or split.
- f_k would be a similar sum over trajectories, but these trajectories would allow k stitchings (loops merging to form new loops).
- In the absence of stitchings, the surface traced out by a trajectory has genus zero.
- k stitchings would allow surfaces of genus k.

Summary

- Quantum Yang–Mills theories are the building blocks of the Standard Model of quantum mechanics. Mathematically undefined.
- Lattice gauge theories are mathematically well-defined discrete approximations of quantum Yang–Mills theories.
- Wilson loop variables are the main objects of interest in lattice gauge theories.
- Gauge-string duality is the phenomenon discovered by theoretical physicists that in the so-called large N limit of special classes of gauge theories, expectations of Wilson loop variables can be written as sums over trajectories in certain kinds of string theories.
- Main result: Expectations of Wilson loop variables in strongly coupled $SO(N)$ lattice gauge theory can indeed be written as sums over trajectories in a lattice string theory.
- Preprint on arXiv.