Spin glass phase at zero temperature in the Edwards–Anderson model

Sourav Chatterjee

What is a spin glass?

- Spin glasses are magnetic materials with strange, “glassy” properties.
- Some common materials that exhibit spin glass behavior include certain types of alloys, such as AuFe and CuMn.
- One of the characteristic properties is the possession of multiple states with near-minimal energies.
- Mathematical models of spin glasses have proved to be very difficult to analyze, perhaps reflecting the complexities of the materials themselves.

Models

- Spin glass models fall broadly into two categories — mean-field models, such as the Sherrington–Kirkpatrick (SK) model, or the more "realistic" lattice models, such as the Edwards–Anderson (EA) model.
- Tremendous progress has been made in the analysis of mean-field models, such as Talagrand's proof of the Parisi formula and Panchenko's proof of ultrametricity.
- The Edwards–Anderson model, on the other hand — as well as other models of spin glasses on a lattice — remain largely intractable.
- Even physicists are not unanimous about the true nature of lattice spin glasses.
- In particular, there is not even a consensus (let alone proof) on whether the EA model at all has the features of a spin glass in some regime of temperature and dimension.

The Edwards–Anderson model

- Take any $d \geq 1$, $L \geq 1$, and let $V = \{0, 1, \ldots, L\}^d$.
- Let E be the set of nearest-neighbor edges of V.
- Let $J = (J_e)_{e \in E}$ be a collection of i.i.d. random variables, which we take to be standard Gaussian in this talk.
- The Edwards–Anderson Hamiltonian (with free boundary condition) in the environment (or disorder) J is the random function $H_J : \{-1, 1\}^V \to \mathbb{R}$ defined as

$$H_J(\sigma) := -\sum_{\{i,j\} \in E} J_{ij} \sigma_i \sigma_j.$$

- A ground state is a state σ (depending on J) that minimizes $H_J(\sigma)$.
- With probability one, there are exactly two ground states, σ and $-\sigma$. The pair $(\sigma, -\sigma)$ is the unique ground state pair.
- We can also consider some given boundary condition (independent of J) that fixes the spins on the boundary of V. Or, we can consider the periodic boundary condition, which identifies opposite faces of V.

Available results about the ground state

- Aizenman and Wehr (1990) proved that the fluctuations of the ground state energy are of order $L^{d/2}$ (both upper and lower bounds).
- A standard approach in mathematical physics is to consider infinite volume ("$L = \infty$") ground states.
- The notion of "minimizing the energy" no longer makes sense in infinite volume, but the difference between the energies of two states that differ only at a finite number of sites is well-defined and finite.
- An infinite volume state is called a ground state if overturning any finite number of spins results in an increase in the energy.
- Aizenman and Wehr (1990) proved that one can construct a measurable map that takes the environment to a probability measure on the set of ground states. Such a map is called a metastate in this literature.

Available results about the ground state (contd.)

- For the last thirty years, the mathematical literature on the Edwards–Anderson model has focused mainly on the study of metastates.
- One of the main goals has been to show that in 2D, there is exactly one ground state pair in infinite volume, with probability one. (Physicists are not unanimous about this.)
- Using a result of Newman and Stein (2001), Arguin, Damron, Newman and Stein (2010) proved that a certain metastate on the 2D half-plane is supported on a unique ground state pair with probability one.
- This is more or less the state of the art in this area. In other words, we understand very little (rigorously) about the Edwards–Anderson model.
- In particular, there is no proof that the EA model exhibits any kind of "spin glass behavior" — that is, different than a ferromagnet.

A marker of "spin glass behavior"

- Consider the Ising ferromagnet on $V = \{0, 1, \ldots, L\}^d$, where $J_{ij} = 1$ for $\{i, j\} \in E$. The ground state pair is $(\sigma, -\sigma)$, where $\sigma_i = 1$ for all i.
- Take a region $A \subseteq V$, whose size is of order L^d.
- Suppose we overturn all spins in A, in the ground state σ.
- This increases the energy by $|\partial A|$, where ∂A is the edge-boundary of A, i.e., the set of all edges from A to $V \setminus A$.
- Physicists tell us that in spin glasses, it is possible to overturn a macroscopic region of spins with energy cost that is negligible compared to the size of the boundary. This is the multiple valley picture for the EA model.
- For example, there are competing claims, made via numerical studies, that in 3D, the energy cost can be as small as $O(L^{1/5})$ (Bray and Moore, 1987), or $O(1)$ (Krzakala and Martin, 2001). Note that $|\partial A|$ is at least of order L^2 in 3D.
- Main difficulty: Finding the ground state is an NP-hard problem. Simulations can be carried out with only rather small values of L, such as $L = 20$.

Energy cost of overturning spins in a given region

- The physics explanation for the existence of low energy excitations in spin glasses is simply "cancellations".
- However, this is not enough justification. A few years ago, I proved (but did not publish) the following theorem.

Theorem (C., 2023)

There are positive constants C_1, C_2 and C_3 depending only on d, such that for any $A \subset V$,

$$\mathbb{P}\left(\frac{\Delta(A)}{|\partial A|} < C_1\right) \le C_2 e^{-C_3|\partial A|},$$

where $\Delta(A)$ is the energy cost of overturning all spins in A in the ground state.

Energy cost of overturning spins in a given region (contd.)

- Thus, for a given region A, overturning the spins in A incurs an energy cost of order $|\partial A|$ (with probability extremely close to 1), just like in ferromagnets.
- So, even if the multiple valley picture holds, it holds only for exceptional regions of overturned spins.
- Given the theorem in the previous slide, it is not inconceivable that such exceptional regions do not actually exist, at least in certain dimensions or certain graphs.
- I spent several years fruitlessly trying to prove (and occasionally, disprove) that such regions do not exist, before realizing, very recently, how to prove that they do exist!

Multiple valleys in the EA model

Theorem (C., 2023)

Take any $d \geq 1$, $L \geq 1$, *and let* $V := \{0, 1, \ldots, L\}^d$. *Let*

$$F := \min\left\{\frac{\Delta(A)}{|\partial A|} : A \subseteq V, \frac{L^d}{4} \leq |A| \leq \frac{3L^d}{4}\right\},$$

where $\Delta(A)$ *is the energy cost of overturning all spins in* A *in the ground state of the EA model. Then for any given boundary condition,*

$$\mathbb{E}(F) \leq \begin{cases} C(d)L^{-1/4}\sqrt{\log L} & \text{if } d = 1, \\ C(d)L^{-1/3}\sqrt{\log L} & \text{if } d \geq 2. \end{cases}$$

For free or periodic boundary,

$$\mathbb{E}(F) \leq C(d)L^{-d/(2d+2)}\sqrt{\log L} \quad \text{for all } d \geq 1.$$

This is the first proof of spin glass behavior (of any kind) in the EA model.

Disorder chaos

- Another physics conjecture about the ground state of the EA model is that it is sensitive to small changes in the disorder J, a phenomenon that is sometimes called disorder chaos.
- Let us consider two kinds of perturbations, both determined by a parameter $p \in (0, 1)$.
- In the first kind of perturbation, we replace each J_e by $(1 - p)J_e + \sqrt{2p - p^2}J'_e$, where $J' = (J'_e)_{e \in E}$ is another set of i.i.d. standard Gaussian random variables, independent of J.
- The coefficients in front of J_e and J'_e are chosen to ensure that the linear combination is again a standard Gaussian random variable.
- In the second kind of perturbation, each J_e is replaced by J'_e with probability p, independently of each other.
- In both cases, the perturbation is small if p is small.

Site overlap

- Let σ be the ground state in the original environment and σ' be the ground state in the perturbed environment.
- The site overlap between the two configurations is defined as

$$R(p) := \frac{1}{|V|}\sum_{i\in V}\sigma_i\sigma_i'.$$

- The site overlap is said to be chaotic with respect to perturbations in the disorder if $R(p) \approx 0$ with high probability for some $p \approx 0$.
- Disorder chaos in the EA model was first conjectured in Fisher and Huse (1986) and Bray and Moore (1987), and verified in various numerical studies since then.

Disorder chaos in the EA model

Theorem (C., 2023)

For $V = \{0, 1, \ldots, L\}^d$ with any given disorder-independent boundary condition, we have that for both kinds of perturbations, for all $p \in (0,1)$,

$$\mathbb{E}(R(p)^2) \le \begin{cases} C(d)L^{-1}p^{-1} & \text{if } d = 1, \\ C(d)L^{-2}p^{-2} & \text{if } d \ge 2. \end{cases}$$

For free or periodic boundary,

$$\mathbb{E}(R(p)^2) \le \frac{C(d)}{L^d p^d} \quad \text{for all } d \ge 1.$$

Thus, $R(p) \approx 0$ with high probability if $p \gg L^{-1}$.

Proof idea

- In the physics literature, usually the multiple valley picture is taken as given, and then chaos is argued to be a consequence of multiple valleys.
- I will go in the opposite direction, by first proving chaos and then deducing multiple valleys as a consequence of chaos.

Simple case: Chaos in 1D

- Consider the EA model on $\{0, 1, \ldots, L\}$, with boundary condition $\sigma_0 = 1$.
- Then the ground state is explicitly given by

$$\sigma_i = \operatorname{sign}(J_{0,1})\operatorname{sign}(J_{1,2}) \cdots \operatorname{sign}(J_{i-1,i}).$$

- Suppose we replace each J_{ij} by $J_{ij} + J_0 K_{ij}$, where K_{ij} is an independent standard Gaussian random variable and J_0 is a small constant.
- The above representation shows that if i is large, then the sign of σ_i becomes unpredictable after this perturbation, since a large number of the couplings in the product on the right will change sign, and this is number is even or odd with approximately equal probability.
- This argument was made precise in Bray and Moore (1987), with explicit estimates.

Generalizing to higher dimensions

- The Bray–Moore argument has no simple generalization to higher dimensions, since there is no explicit formula for the ground state in $d \geq 2$. That is why this and other questions about the EA model have remained unsolved for 35 years.
- I will now show that the argument generalizes in principle.
- Take any i and j, and consider $\sigma_i \sigma_j$ (where σ is the ground state) as a function of the couplings (J_{kl}).
- Unlike in 1D, we do not expect this to be a function of solely the signs of the J_{kl}'s.
- However, it can certainly be expanded as a multivariate Hermite polynomial series, since the multivariate Hermite polynomials form an orthonormal basis of L^2 functions of independent Gaussian random variables.
- I will now show that *all terms in this expansion have degree greater than or equal to the distance between i and j.* As in 1D, this will lead to a proof of chaos.

Hermite polynomial expansion

- Let $h_0, h_1, \ldots$ be the orthonormal basis of normalized Hermite polynomials for $L^2(\mu)$, where μ is the standard Gaussian distribution on $\mathbb{R}$ and $h_0 \equiv 1$.
- Then an orthonormal basis of $L^2(J)$ is formed by products like $h_n(J) := \prod_{e\in E} h_{n_e}(J_e)$, where $n_e \in \mathbb{N} := \{0, 1, \ldots\}$ for each e, and $n := (n_e)_{e\in E} \in \mathbb{N}^E$.
- Any square-integrable function $f(J)$ of the disorder J can be expanded in this basis as

$$f(J) = \sum_{n\in\mathbb{N}^E} \hat{f}(n) h_n(J),$$

  where

$$\hat{f}(n) := \mathbb{E}(f(J)h_n(J)).$$

- The infinite series on the right side in should be interpreted as the L^2-limit of partial sums, where the order of summation is irrelevant.

Main lemma

- Take any distinct $i, j \in V$.
- Let σ be the ground state. Consider $\sigma_i \sigma_j$ as a function $\phi(J)$ of the disorder J.
- For any $n \in \mathbb{N}^E$, let $E(n)$ be the set of edges $e \in E$ such that $n_e > 0$, and $V(n)$ be the set of vertices that are endpoints of the edges in $E(n)$. Let $G(n)$ be the graph $(V(n), E(n))$.
- The following lemma is the main ingredient for the proof of chaos.

Lemma (C., 2023)

Let all notations be as above. Then $\hat{\phi}(n) = 0$ unless both i and j are in $V(n)$ and the connected components of $G(n)$ containing i and j are either the same, or both intersect the boundary ∂V (in case we have a boundary condition).

Proof sketch for lemma

- For simplicity, consider the case of free boundary.
- Also, for simplicity, let us work under the assumption that $i, j \in V(n)$.
- Let us show that i, j are in the same connected component of $G(n)$.
- Suppose not. Then there is a contour γ separating the components containing i and j, such that no edge in γ is in $E(n)$.
- Suppose we replace J_e by $-J_e$ for each $e \in \gamma$.
- Then $\sigma_i\sigma_j$ changes to $-\sigma_i\sigma_j$. This shows that

$$\mathbb{E}(\sigma_i\sigma_j | (J_e)_{e\notin\gamma}) = -\mathbb{E}(\sigma_i\sigma_j | (J_e)_{e\notin\gamma}),$$

  and thus, $\mathbb{E}(\sigma_i\sigma_j | (J_e)_{e\notin\gamma}) = 0$.
- But $h_n(J)$ does not depend on $(J_e)_{e\in\gamma}$. Thus,

$$\hat{\phi}(n) = \mathbb{E}(\sigma_i\sigma_j h_n(J)) = \mathbb{E}(\mathbb{E}(\sigma_i\sigma_j | (J_e)_{e\notin\gamma}) h_n(J)) = 0.$$

Main consequence of lemma

- Recall: We have fixed $i, j \in V$, and considered $\sigma_i \sigma_j$ as a function $\phi(J)$ of the disorder J.
- For $n \in \mathbb{N}^E$, $\hat{\phi}(n)$ denotes the coefficient of $h_n(J)$ in the Hermite polynomial expansion of $\phi(J)$.
- For each n, $E(n)$ denotes the set of all e such that $n_e > 0$, and $V(n)$ is the set of endpoints of these edges.
- The lemma shows that $\hat{\phi}(n) = 0$ unless both i and j are in $V(n)$ and the connected components of $G(n)$ containing i and j are either the same, or both intersect the boundary ∂V.
- From this, we conclude that

  $$|E(n)| \geq \min\{d(i,j), d(i,\partial V) + d(j,\partial V)\},$$

  where d denotes graph distance on V.
- Thus, the Hermite polynomial expansion of $\sigma_i \sigma_j$ consists only of terms that are products of at least $\min\{d(i,j), d(i,\partial V) + d(j,\partial V)\}$ univariate Hermite polynomials.

Proving chaos: Step 1

- Let $J(p)$ be obtained from J by applying a perturbation of size $p \in (0,1)$, of either kind.
- Then for any e and any $n \in \mathbb{N} \setminus \{0\}$,

$$\mathbb{E}(h_n(J_e(p))|J_e) = \begin{cases} (1-p)^n h_n(J_e) & \text{for perturbation of } 1^{\text{st}} \text{ kind,} \\ (1-p) h_n(J_e) & \text{for perturbation of } 2^{\text{nd}} \text{ kind.} \end{cases}$$

- Thus, for any $n \in \mathbb{N}^E$,

$$\mathbb{E}(h_n(J(p))|J) = \begin{cases} (1-p)^{\sum_e n_e} h_n(J) & \text{for perturbation of } 1^{\text{st}} \text{ kind,} \\ (1-p)^{|E(n)|} h_n(J) & \text{for perturbation of } 2^{\text{nd}} \text{ kind.} \end{cases}$$

(Recall that $E(n) = \{e : n_e > 0\}$.)

Proving chaos: Step 2

- Let $\sigma(p)$ be the ground state after perturbation. Then by the previous slide,

$$\mathbb{E}(\sigma_i(p)\sigma_j(p)|J) = \sum_n \hat{\phi}(n)\mathbb{E}(h_n(J(p))|J)$$
$$= \begin{cases} \sum_n (1-p)^{\sum_e n_e}\hat{\phi}(n)h_n(J) & \text{for perturbation of 1}^{\text{st}}\text{ kind,} \\ \sum_n (1-p)^{|E(n)|}\hat{\phi}(n)h_n(J) & \text{for perturbation of 2}^{\text{nd}}\text{ kind.} \end{cases}$$

- But by the main lemma, whenever $\hat{\phi}(n) \neq 0$,

$$\sum_e n_e \geq |E(n)| \geq m := \min\{d(i,j), d(i,\partial V) + d(j,\partial V)\}.$$

- Thus, by the Parseval identity for the Hermite polynomial expansion,

$$\mathbb{E}[(\mathbb{E}(\sigma_i(p)\sigma_j(p)|J))^2] \leq (1-p)^{2m}\sum_n \hat{\phi}(n)^2$$
$$= (1-p)^{2m}\mathbb{E}((\sigma_i\sigma_j)^2) = (1-p)^{2m}.$$

Proof of chaos: Final step

- Note that

$$\begin{aligned}|\mathbb{E}(\sigma_i\sigma_j\sigma_i(p)\sigma_j(p))| &= |\mathbb{E}[\sigma_i\sigma_j\mathbb{E}(\sigma_i(p)\sigma_j(p)|J)]| \\ &\le \mathbb{E}|\mathbb{E}(\sigma_i(p)\sigma_j(p)|J)| \\ &\le \sqrt{\mathbb{E}[(\mathbb{E}(\sigma_i(p)\sigma_j(p)|J))^2]}.\end{aligned}$$

- Combining with the previous slide, we get

$$|\mathbb{E}(\sigma_i\sigma_j\sigma_i(p)\sigma_j(p))| \le (1-p)^{\min\{d(i,j),d(i,\partial V)+d(j,\partial V)\}}.$$

- This gives a mathematical proof of the observation of Bray and Moore (1987) that "relative orientations of spins with large separations are sensitive to small changes in the bond strengths".
- It is now easy to complete the proof of chaos, since

$$R(p)^2 = \frac{1}{|V|^2}\sum_{i,j\in V}\sigma_i\sigma_j\sigma_i(p)\sigma_j(p).$$

From chaos to multiple valleys

- Let A be the region that is overturned when we apply a perturbation of the first kind of size p, where $L^{-1} \ll p \ll 1$.
- By the chaos theorem, we know that $R(p) \approx 0$. This implies that $|A| \approx L^d/2$.
- We will show that the cost of overturning the spins in A is negligible compared to the size of ∂A.

From chaos to multiple valleys (contd.)

- Since $\sigma(p)$ minimizes $H_{J(p)}$, $H_{J(p)}(\sigma) - H_{J(p)}(\sigma(p)) \geq 0$.
- But also,

$$\begin{aligned} H_{J(p)}(\sigma) - H_{J(p)}(\sigma(p)) &= -2 \sum_{\{i,j\}\in\partial A} J_{ij}(p)\sigma_i\sigma_j \\ &= -2(1-p) \sum_{\{i,j\}\in\partial A} J_{ij}\sigma_i\sigma_j - 2\sqrt{2p-p^2} \sum_{\{i,j\}\in\partial A} J'_{ij}\sigma_i\sigma_j \\ &= -(1-p)(H_J(\sigma(p)) - H_J(\sigma)) - 2\sqrt{2p-p^2} \sum_{\{i,j\}\in\partial A} J'_{ij}\sigma_i\sigma_j. \end{aligned}$$

- Combining the last two observations, we get

$$H_J(\sigma(p)) - H_J(\sigma) \leq C\sqrt{p}|\partial A| \max_{\{i,j\}\in E} |J'_{ij}|.$$

- Since $\max_{\{i,j\}\in E} |J'_{ij}| = O(\sqrt{\log L})$, we can now complete the proof by taking $p = L^{-\alpha}$ for some suitable $\alpha < 1$.

Main open question

- There are many open questions, but the most important one is the following.
- In 3D, does there exist a macroscopic region where the spins in the ground state can be overturned with an energy cost of $O(1)$ as $L \to \infty$?
- If true, this would support the Parisi picture of lattice spin glasses.
- If false (i.e., if the minimum energy cost grows with L), that would support the Fisher–Huse droplet theory.
- The validity of the Parisi picture versus the droplet theory (for lattice spin glasses) is arguably the most important open problem about spin glasses at present.

A related question

- We have seen that $\mathbb{E}(\sigma_i(p)\sigma_j(p)|J)$ drop sharply to zero as p increases from 0 to a small positive value, if i and j are far apart.
- Suppose i and j are neighbors. Then one can show that $\mathbb{E}(\sigma_i(p)\sigma_j(p)|J)$ will not drop to zero — but does it drop sharply to a value less than 1?
- More precisely, is it true that under periodic boundary condition, for neighboring i and j,

$$\lim_{p\to 0}\lim_{L\to\infty}\mathbb{E}[(\mathbb{E}(\sigma_i(p)\sigma_j(p)|J))^2] < 1?$$

- This, too, would settle the debate between the Parisi picture versus the droplet theory. (The Parisi picture holds if the above is true; the droplet theory, if not.)