Rigidity of the 3D hierarchical Coulomb gas

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Rigidity of point processes

- Let P be a Poisson point process of intensity n in $\mathbb{R}^d$, and let $A \subseteq \mathbb{R}^d$ be a set of nonzero volume.
- Let $N(A) := |A \cap P|$.
- Then $\mathbb{E}(N(A)) = \mathrm{Var}(N(A)) = \mathrm{vol}(A)n$.
- Thus, $N(A)$ has fluctuations of order $\sqrt{n}$.
- Let Q be another point process with n particles per unit volume on average.
- If $|Q \cap A|$ has fluctuations of order $o(\sqrt{n})$, it is called rigid. (This is one definition; there are others.)
- Called hyperuniformity in the physics literature.

Examples of rigid point processes

- Many examples, intensely studied in recent years.
- Eigenvalues of random matrices, Coulomb gas and other interacting gases, zeros of random analytic functions, determinantal point processes, orthogonal polynomial ensembles, etc.
- Numerous contributors:
  - **Eigenvalues and determinantal processes:** Borodin, Bourgade, Deift, Diaconis, Erdős, Evans, Forrester, Guionnet, Johansson, Pastur, Rider, Shcherbina, Soshnikov, Tao, Virág, Vu, Yau, ...
  - **Coulomb gas and other interacting gases:** Bauerschmidt, Ben Arous, Bourgade, Chafaï, Leblé, Majumdar, Radin, Serfaty, Yau, Zeitouni, ...
  - **Zeros of random analytic functions:** Ghosh, Lebowitz, Nazarov, Peres, Sodin, Volberg, ...
  - **Orthogonal polynomial ensembles:** Bardenet, Berman, Breuer, Duits, Hardy, Johansson, Lambert, ...
  - $+$ many others (see my preprint on arXiv for a survey).

Interacting gases

- Consider a probability density on $(\mathbb{R}^d)^n$ of the form

$$\frac{1}{Z}\exp\biggl(-\beta\sum_{1\le i<j\le n} w(x_i,x_j) - \beta n\sum_{i=1}^n V(x_i)\biggr),$$

  where w is a symmetric function, V is any function with sufficient growth at infinity, β is the "inverse temperature" parameter, and Z is the normalizing constant.
- This is the general form of an interacting gas of n particles with pairwise interactions.
- Coulomb gas: V is arbitrary (usually $V(x) = |x|^2$) and

$$w(x,y) = \begin{cases} |x-y| & \text{if } d = 1,\\ -\log|x-y| & \text{if } d = 2,\\ |x-y|^{2-d} & \text{if } d \ge 3.\end{cases}$$

- Log gas: $d = 1$, $w(x,y) = -\log|x-y|$ and $V(x) = x^2$.

Interacting gases and random matrices

- One key motivation for studying interacting gases comes from random matrix theory.
- Eigenvalues of GUE, GOE and unitary random matrices are 1D log gases.
- Eigenvalues of Ginibre random matrices are 2D Coulomb gases.
- The 1D log gas and the 1D and 2D Coulomb gases are known to be rigid (more later).
- However, the most physically relevant interacting gas is the 3D Coulomb gas. No connection with random matrices or determinantal point processes. Very few rigorous mathematical results are known about it. In particular, it is believed to be rigid but there is no rigorous proof.

Rigidity of interacting gases

- Rigidity of the 1D log gas follows from the works of many authors, e.g. Costin & Lebowitz (1995), Diaconis & Evans (2001), Wieand (2002), Pastur (2006), Bourgade, Erdős & Yau (2012) and Tao & Vu (2013).
- For the 2D Coulomb gas with $V(x) = |x|^2$, various forms of rigidity were established by Borodin & Sinclair (2009), Bourgade, Yau & Yin (2014), Tao & Vu (2015), and Ghosh & Peres (2017).
- For the 2D Coulomb gas with general V, rigidity was recently established through contributions from Sandier & Serfaty (2015), Rougerie & Serfaty (2015), Bauerschmidt, Bourgade, Nikula & Yau (2016) and Leblé & Serfaty (2016).

Interacting gases in three and higher dimensions

- Rigidity of the Coulomb gas has not yet been proved in dimensions three and higher.
- The most promising results available at this time are due to Serfaty and collaborators, who have obtained very precise informations about normalizing constants and large deviations for Coulomb gases in general dimensions.

The prediction of Jancovici, Lebowitz and Manificat

- Consider a 3D Coulomb gas of n particles.
- Let $N(A)$ be the number of particles falling in a region A of nonzero volume.
- In 1993, Jancovici, Lebowitz and Manificat made a famous prediction (with a physics justification) that $N(A)$ should have fluctuations of order $n^{1/3}$.
- This is much larger than similar fluctuations for the 1D log gas (arising in random matrices), which are of order $\sqrt{\log n}$.
- For the 3D Coulomb gas, however, this conjecture is still open.
- In a recent preprint, I proved this conjecture (up to logarithmic factors) for a closely related model, known as the 3D hierarchical Coulomb gas. This is the subject of this talk.

The 3D hierarchical Coulomb gas

- Recall the general form of the probability density for an interacting gas of n particles:

$$\frac{1}{Z}\exp\biggl(-\beta\sum_{1\le i<j\le n} w(x_i,x_j)-\beta n\sum_{i=1}^{n}V(x_i)\biggr).$$

- We will take $d = 3$, and

$$V(x)=\begin{cases}0 & \text{if } x\in[0,1]^3,\\ \infty & \text{if } x\notin[0,1]^3,\end{cases}$$

  so that all particles are confined inside the unit cube.
- Finally, let $w(x,y)=2^{k(x,y)}$, where $k(x,y)$ is the minimum k such that x and y belong to different dyadic cubes of side-length 2^{-k}.
- Then $w(x,y)$ "behaves like" the 3D Coulomb potential $|x-y|^{-1}$ when x is close to y, up to constant factors.

Dyson's hierarchical scheme

- The scheme described in the previous slide is a general method for replacing ordinary Euclidean distance by a hierarchical version of itself, originally proposed by Dyson (1953) and used subsequently in thousands of papers.
- The hierarchical version of the Coulomb gas was introduced in the physics literature by Benfatto, Gallavotti & Nicolò (1986) and subsequently studied by many authors.

Main result: Upper bound

Theorem (C., 2017)

Consider the n-particle 3D hierarchical Coulomb gas in the unit cube. Take any $A \subseteq [0,1]^3$ with a two-dimensional boundary (in the Minkowski sense) and let $N(A)$ be the number of particles falling in A. Then $\mathbb{E}(N(A)) = \mathrm{vol}(A)n$ and

$$\mathrm{Var}(N(A)) \le Cn^{2/3} \log n,$$

where C is a constant that depends only on A and β.

(There is another similar result in the preprint for the case where A is shrinking with n. Rigidity is proved at all scales.)

Main result: Lower bound

Theorem (C., 2017)

Suppose that A *is nonempty, connected and open, and* ∂A *is a smooth, closed, orientable surface. Then there exist* $n_0 \geq 1$, $c_1 > 0$ *and* $c_2 < 1$, *depending only on* A *and* β, *such that for any* $n \geq n_0$ *and any* $-\infty < a \leq b < \infty$ *with* $b - a \leq c_1 n^{1/3}$,

$$\mathbb{P}(a \leq N(A) \leq b) \leq c_2.$$

What causes rigidity in interacting gases?

- There is a folklore that "repulsion" between points causes rigidity, as in determinantal processes.
- The reasoning is not clear to me. In fact, I do not know of any example where one gets a proof of rigidity starting with this intuition.
- A different intuitive explanation is that rigid point processes behave essentially like a perturbed lattice, where each point is a small perturbation of a corresponding deterministic value.
- This can be partly validated for eigenvalues of Hermitian random matrices, where we can talk about the k^{th} largest eigenvalue and its fluctuations, for $k = 1, 2, \ldots, n$.
- However, it is not clear how to make use of this intuition to construct proofs in higher dimensions.
- I will now give a different intuition, using a toy example involving balls and boxes. The proof for the 3D hierarchical Coulomb gas is a generalization of this toy proof.

Interacting balls and boxes

- Suppose that we have two boxes, and $2n$ balls are to be dropped at random into these two boxes.
- Two balls falling into the same box contribute a quantity a to the potential energy, and two balls falling in different boxes contribute a quantity b to the potential energy.
- Thus, if n_1 and n_2 are the numbers of balls falling in boxes 1 and 2, then the total potential energy is

$$H(n_1, n_2) := \binom{n_1}{2} a + \binom{n_2}{2} a + n_1 n_2 b.$$

- A configuration with n_1 balls in box 1 and n_2 balls in box 2 is assigned a probability proportional to $e^{-\beta H(n_1,n_2)}$, where β is the inverse temperature parameter, as usual.

Fluctuations in the toy model

- Let N_1 and N_2 be the numbers of balls in boxes 1 and 2 in a random configuration drawn from this model. Note that $N_1 + N_2 = 2n$.
- Question: What is the order of fluctuations of N_1?
- If $a = b$, then the balls are distributed uniformly at random between the two boxes. In this case, N_1 has fluctuations of order $\sqrt{n}$.
- What if $a > b$?
- Answer: If $a > b$, then N_1 has fluctuations of order 1 as $n \to \infty$.
- Let me now explain how to see this.

Rigidity in interacting balls and boxes

- For each n_1, n_2 such that $n_1 + n_2 = 2n$, let

$$Z(n_1, n_2) := Q(n_1, n_2)e^{-\beta H(n_1,n_2)},$$

where $Q(n_1, n_2)$ is the number of configurations that have n_1 balls in box 1 and n_2 balls in box 2.

- Explicitly,

$$Q(n_1, n_2) = \frac{(2n)!}{n_1!n_2!}.$$

- Then note that the normalizing constant for the toy model is

$$Z = \sum_{n_1,n_2\,:\,n_1+n_2=2n} Z(n_1, n_2).$$

- Moreover, for any n_1, n_2 such that $n_1 + n_2 = 2n$,

$$\mathbb{P}(N_1 = n_1,\ N_2 = n_2) = \frac{Z(n_1, n_2)}{Z}.$$

Rigidity in interacting balls and boxes, contd.

- Recall:

$$H(n_1, n_2) = \binom{n_1}{2} a + \binom{n_2}{2} a + n_1 n_2 b.$$

- A simple calculation gives

$$H(n+k, n-k) = H(n, n) + k^2(a-b).$$

- Also, recall that $Z(n_1, n_2) = Q(n_1, n_2)e^{-\beta H(n_1, n_2)}$, where

$$Q(n_1, n_2) = \frac{(2n)!}{n_1! n_2!}.$$

- Not hard to see that

$$\frac{Q(n+k, n-k)}{Q(n, n)} \sim e^{-k^2/2n}.$$

- Thus, if $k \ll \sqrt{n}$, then

$$\frac{Z(n+k, n-k)}{Z(n, n)} \sim e^{-\beta k^2 (a-b) + o(1)}.$$

Rigidity in interacting balls and boxes, contd.

- This gives, for $k \ll \sqrt{n}$,

$$\begin{aligned}&\mathbb{P}(N_1 = n+k,\ N_2 = n-k)\\&\le \frac{\mathbb{P}(N_1 = n+k,\ N_2 = n-k)}{\mathbb{P}(N_1 = N_2 = n)}\\&= \frac{Z(n+k, n-k)}{Z(n,n)} \sim e^{-\beta k^2(a-b)+o(1)}.\end{aligned}$$

- The case $k \ge \sqrt{n}$ is simpler and may be dealt with separately. This proves $O(1)$ fluctuations for N_1 around its expected value n.
- The above argument generalizes to any finite number of boxes, as long as the matrix of potentials is strictly positive definite.

A general heuristic

- The toy model suggests the following general heuristic for rigidity of interacting gases:
  - Suppose that the potential is positive definite, and consider an energy minimizing configuration.
  - Shifting a few points around increases the energy, but the increase is not adequately compensated by the variation in the entropy (combinatorial) term.
  - Thus, it is unlikely that the system deviates far from the energy minimizing state.
- This looks similar to the usual energy-entropy argument of statistical mechanics, but there is an important difference.
- Here, we inspect the *changes in energy and entropy due to small changes in the ground state*. The calculations become more delicate.
- In some sense, this is a combination of energy-entropy competition and the cavity method.

Back to the 3D hierarchical Coulomb gas

- Let $Z(n)$ be the partition function of the n-particle 3D hierarchical Coulomb gas.
- Using the previous heuristic, the question can be boiled down to understanding the ratio $Z(n+1)/Z(n)$. (This is reminiscent of the cavity method of spin glasses.)
- On the one hand,

$$\frac{Z(n+1)}{Z(n)} = \mathbb{E} \exp\biggl(-\beta \sum_{i=1}^{n} w(U, X_i)\biggr),$$

where $(X_1, \ldots, X_n)$ is a realization of the n-particle system, and $U \sim Unif[0,1]^d$.

- Jensen's inequality gives $Z(n+1)/Z(n) \geq e^{-\beta \alpha n}$, where $\alpha = \iint w(x,y)dxdy$, since each X_i is uniformly distributed.

Estimating $Z(n+1)/Z(n)$

- On the other hand,
$$\frac{Z(n)}{Z(n+1)} = \mathbb{E}\exp\left(\beta\sum_{i=1}^{n} w(X_{n+1}, X_i)\right),$$
where $(X_1, \ldots, X_{n+1})$ is a realization of the $(n+1)$-particle system.
- Again, Jensen's inequality gives
$$\frac{Z(n)}{Z(n+1)} \ge \exp\left(\beta\sum_{i=1}^{n} \mathbb{E}(w(X_{n+1}, X_i))\right).$$
- But by the symmetry between $X_1, \ldots, X_{n+1}$, this equals
$$\exp\left(\frac{\beta n}{\binom{n+1}{2}} \sum_{1\le i<j\le n+1} \mathbb{E}(w(X_i, X_j))\right)$$
$$= \exp\left(\frac{\beta n}{\binom{n+1}{2}} \mathbb{E}(H_{n+1}(X_1, \ldots, X_{n+1}))\right).$$

Estimating $Z(n+1)/Z(n)$, contd.

- Thus,

$$\frac{Z(n)}{Z(n+1)} \geq \exp\left(\frac{\beta n}{\binom{n+1}{2}} L_{n+1}\right),$$

where L_{n+1} is the minimum possible energy of the $(n+1)$-particle system.

- We will now show that

$$L_{n+1} \geq \binom{n+1}{2}\alpha - Cn^{4/3},$$

where C is a universal constant and $\alpha = \iint w(x,y)dxdy$, as before.

- Combined with the lower bound on $Z(n+1)/Z(n)$, this shows that

$$e^{-\beta\alpha n} \leq \frac{Z(n+1)}{Z(n)} \leq e^{-\beta\alpha n + Cn^{1/3}}.$$

Using $Z(n+1)/Z(n)$

- Suppose that we have a configuration where each of the 8 sub-cubes of $[0,1]^3$ receive $n/8$ particles.
- These estimates for $Z(n+1)/Z(n)$ show that moving k particles from one box to another increases the energy of the configuration by Ck^2 while increasing the entropy by at most $Ckn^{1/3}$.
- This shows that the chance of having more than $n/8 + O(n^{1/3})$ particles in any box is small.
- A multi-scale generalization of this argument leads to the proof.

Lower bound on the ground state energy

- We wish to show that

$$L_n \geq \binom{n}{2}\alpha - Cn^{4/3}$$

and $\alpha = \iint w(x,y)dxdy$.

- Let $\mathcal{D}_j$ be the set of dyadic cubes of side-length 2^{-j}.
- Take any configuration $(x_1, \ldots, x_n)$.
- For each dyadic cube D, let n_D be the number of points falling in D.
- The energy of the 3D hierarchical Coulomb gas can be written as a multi-scale χ^2 statistic:

$$H_n(x_1, \ldots, x_n) = \sum_{j=1}^{\infty} \sum_{D \in \mathcal{D}_j} 2^j \binom{n_D}{2} + 2\binom{n}{2}.$$

Lower bound on the ground state energy, contd.

- Thus, for any k,

$$H_n(x_1, \ldots, x_n) \ge \sum_{j=1}^{k} \sum_{D \in \mathcal{D}_j} 2^j \binom{n_D}{2} + 2\binom{n}{2}$$
$$= \sum_{j=1}^{k} \sum_{D \in \mathcal{D}_j} 2^{j-1} n_D^2 - \sum_{j=1}^{k} 2^{j-1} n + 2\binom{n}{2}.$$

- By the Cauchy–Schwarz inequality,

$$\sum_{D \in \mathcal{D}_j} n_D^2 \ge \frac{n^2}{8^j}.$$

- Thus,

$$H_n(x_1, \ldots, x_n) \ge \frac{n^2}{2} \sum_{j=1}^{k} 4^{-j} - 2^k n + 2\binom{n}{2}.$$

Lower bound on the ground state energy, contd.

- This gives

$$H_n(x_1, \ldots, x_n) \geq \frac{7}{3}\binom{n}{2} - \frac{n^2}{6}4^{-k} - 2^k n.$$

- A simple calculation gives $\alpha = 7/3$.
- Choosing k such that $n^{1/3} \leq 2^k \leq 2n^{1/3}$ gives the desired lower bound.

Summary

- A point process is called rigid if the number of points falling in a given set has much smaller fluctuations that the corresponding number for a Poisson point process of the same intensity.
- Many examples, widely studied in recent years.
- Most of the available results are for 1D and 2D processes.
- This talk was about a rigidity result for a three-dimensional interacting gas, known as the 3D hierarchical Coulomb gas. It is a close cousin of the 3D Coulomb gas.
- The main result gives matching upper and lower bounds on the order of fluctuations (up to a logarithmic factor).
- Proof technique is based on a general approach that combines energy-entropy competition and the cavity method.
- The corresponding result for the 3D Coulomb gas, predicted by Jancovici, Lebowitz and Manificat in 1993, remains an open problem.