# A probabilistic mechanism for quark confinement

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# Quantum Yang–Mills theories

- Quantum gauge theories, also known as quantum Yang–Mills theories, are components of the Standard Model of quantum mechanics.
- In spite of many decades of research, physically relevant quantum gauge theories have not yet been constructed in a rigorous mathematical sense.
- The most popular approach to solving this problem is via the program of constructive field theory.
- In this approach, one starts with a statistical mechanical model on the lattice; the next step is to pass to a continuum limit of this model; the third step is to show that the continuum limit satisfies certain 'axioms'; if these axioms are satisfied, then there is a standard machinery which allows the construction of a quantum field theory.
- Taking this program to its completion is one of the Clay millennium problems.

# Lattice gauge theories

- The statistical mechanical models considered in the first step of the above program are known as lattice gauge theories.
- A lattice gauge theory may be coupled with a Higgs field, or it may be a pure lattice gauge theory.
- We will only deal with pure lattice gauge theories in this talk.
- A pure lattice gauge theory is characterized by its gauge group (usually a compact matrix Lie group), the dimension of spacetime, and a parameter known as the coupling strength.
- These theories on their own, even without passing to the continuum limit or constructing the quantum theory, can yield substantial physically relevant information.
- Enormous amount of computational effort is spent in laboratories around the world to numerically predict masses of elementary particles and other quantities using lattice gauge theories.

# Mass gap

- Two very important open questions have lattice gauge theoretic formulations.
- The first is the question of Yang–Mills mass gap.
- In lattice gauge theories, mass gap is equivalent to exponential decay of correlations under a certain kind of boundary condition.
- Mass gap is not hard to establish at sufficiently large values of the coupling strength.
- However, the theories are physically relevant only at weak coupling.
- It is not known how to mathematically establish mass gap in 4D and 3D lattice gauge theories at weak coupling.
- Huge Monte Carlo studies, however, show beyond doubt that the conjecture is correct, and give correct physical predictions.

# Quark confinement

- The second big open question is the problem of quark confinement.
- Quarks are the constituents of various elementary particles, such as protons and neutrons.
- It is an enduring mystery why quarks are never observed freely in nature.
- The problem of quark confinement has received a lot of attention in the physics literature, and yet the current consensus seems to be that a satisfactory theoretical explanation does not exist (let alone a rigorous proof!).
- Wilson (1974) argued that quark confinement is equivalent to showing that the relevant lattice gauge theory satisfies what's now known as Wilson's area law.
- A number of deep results are known about the area law (will discuss soon), but the main question remains open.

# Content of this talk

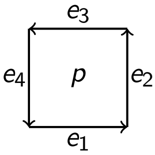
- Main result: *If the gauge group is compact, connected, and has a nontrivial center, then the presence of exponential decay of correlations under arbitrary boundary conditions implies that Wilson's area law holds.*
- The nontriviality of the center is known as center symmetry.
- The exponential decay assumption is stronger than usual mass gap, which means exponential decay under a specific boundary condition. I call it strong mass gap.
- There is a longstanding belief in physics, originating in the work of 't Hooft (1978), that mass gap plus unbroken center symmetry implies confinement. (Note: 'Unbroken center symmetry' is not a rigorously defined concept.)
- The above result shows that *strong mass gap plus center symmetry implies confinement*. The physical explanation (according to Ed Witten, personal communication) is that strong mass gap prevents breaking of center symmetry.

# Notation

- Let $n \geq 1$ and $d \geq 2$ be two integers.
- Let $G$ be a closed connected subgroup of the unitary group $U(n)$.
- Let $B_N := [-N, N]^d \cap \mathbb{Z}^d$.
- Let $E_N$ be the set of positively oriented nearest-neighbor edges of $B_N$.
- Let $\Omega_N$ be the set of all functions from $E_N$ into $G$.
- If $\omega \in \Omega_N$ and $e$ is a negatively oriented edge, we define $\omega_e := \omega_{e^{-1}}^{-1}$, where $e^{-1}$ is the positively oriented version of $e$.

# Plaquettes

- A plaquette in $\mathbb{Z}^d$ is a set of four directed edges that form the boundary of a square.
- Let $P_N$ be the set of all plaquettes in $B_N$.
- Given some $p \in P_N$ and $\omega \in \Omega_N$, we define $\omega_p$ as follows.
- Write $p$ as a sequence of directed edges $e_1, e_2, e_3, e_4$, each one followed by the next.




- Let $\omega_p := \omega_{e_1}\omega_{e_2}\omega_{e_3}\omega_{e_4}$.
- Although there are ambiguities in this definition about the choice of $e_1$ and the direction of traversal, that is not problematic because we will only use the quantity $\Re(\mathrm{Tr}(\omega_p))$, which is not affected by these ambiguities.

# Hamiltonian

- Let $\partial E_N$ denote the set of positively oriented boundary edges of $B_N$.
- Let $\partial \Omega_N$ denote the set of all functions from $\partial E_N$ into $G$.
- An element of $\partial \Omega_N$ will be called a boundary condition.
- Let $E_N^\circ := E_N \setminus \partial E_N$ be the set of positively oriented interior edges of $B_N$.
- Let $\Omega_N^\circ$ be the set of all functions from $E_N^\circ$ into $G$. An element of $\Omega_N^\circ$ will be called a configuration.
- Take any boundary condition $\delta$. For each $\omega \in \Omega_N^\circ$, extend $\omega$ to an element $\widetilde{\omega} \in \Omega_N$ by defining

$$\widetilde{\omega}_e := \begin{cases} \omega_e & \text{if } e \in E_N^\circ, \\ \delta_e & \text{if } e \in \partial E_N. \end{cases}$$

- Define the Hamiltonian

$$H_{N,\delta}(\omega) := \sum_{p \in P_N} \Re(\mathrm{Tr}(\widetilde{\omega}_p)).$$

# Definition of lattice gauge theory

- The pure lattice gauge theory on $B_N$ with gauge group $G$, coupling parameter $\beta$, and boundary condition $\delta$, is the probability measure $\mu_{N,\delta,\beta}$ on $\Omega_N^\circ$ defined as

$$d\mu_{N,\delta,\beta}(\omega) = Z_{N,\delta,\beta}^{-1} e^{\beta H(\omega)} d\lambda_N(\omega),$$

  where $\lambda_N$ is the product Haar measure on $\Omega_N^\circ$ and $Z_{N,\delta,\beta}$ is the normalizing constant.
- Here $\beta = 1/g_0^2$, where $g_0$ is the coupling strength.
- Given a measurable function $f : \Omega_N^\circ \to \mathbb{C}$, the expected value of the function under the above lattice gauge theory is the quantity

$$\langle f \rangle := \int_{\Omega_N^\circ} f(\omega) d\mu_{N,\delta,\beta}(\omega),$$

  provided that the integral on the right is well-defined.

# Area law

- Let $\pi$ be a finite-dimensional unitary representation of the group $G$, and let $\chi_\pi$ be the character of $\pi$.
- Let $\ell$ be a closed loop in $B_N$, with directed edges $e_1, \ldots, e_k$.
- Given a configuration $\omega$, the Wilson loop variable $W_\ell(\omega)$ is defined as

$$W_\ell(\omega) := \chi_\pi(\omega_{e_1}\omega_{e_2}\cdots\omega_{e_k}).$$

- The lattice gauge theory is said to satisfy Wilson's area law for the representation $\pi$ if

$$|\langle W_\ell \rangle| \le C_1 e^{-C_2 \, \mathrm{area}(\ell)}$$

  for any rectangular loop $\ell$, where $C_1$ and $C_2$ are positive constants that depend only on $G$, $\beta$, $d$ and $\pi$, and $\mathrm{area}(\ell)$ is the area enclosed by $\ell$.

# Why does area law imply quark confinement?

- Let $V(R)$ be the potential energy of a static quark-antiquark pair separated by distance $R$.
- QFT calculations imply that for a rectangular loop $\ell$ of side-lengths $R$ and $T$ in the continuum limit of 4D $SU(3)$ theory and a suitable representation $\pi$, $\langle W_\ell \rangle$ should behave like $e^{-V(R)T}$.
- So if the area law holds, then $V(R)$ grows linearly in the distance $R$ between the quark and the antiquark. By the conservation of energy, this implies that the pair will not be able to separate beyond a certain distance.
- Renormalization group arguments predict that $\beta$ has to be sent to infinity as the lattice spacing goes to zero to obtain the continuum limit of 4D non-Abelian theories. This indicates that we need the area law to hold at arbitrarily large values of $\beta$ in 4D $SU(3)$ theory for it to imply confinement of quarks.

# Area law: Basic facts

- It is easy to show that the area law holds at all $\beta$ in any 2D theory, since gauge fixing can be used to reduce any 2D theory to a 1D model.
- Seiler (1978) proved an area law lower bound for any theory at any $\beta$.
- Simon and Yaffe (1982) proved a perimeter law upper bound for any theory at any $\beta$.

# Area law: Deeper results

- Osterwalder and Seiler (1978) showed that the area law holds at small enough $\beta$ (strong coupling) for any theory in any dimension.
- Guth (1980) and Fröhlich and Spencer (1982) showed that for 4D $U(1)$ theory, area law breaks down at large enough $\beta$; instead, perimeter law holds. This is known as the deconfinement transition for this theory.
- The deconfinement transition was physically expected, because 4D $U(1)$ theory is related to photons, which are not confined.
- Göpfert and Mack (1982) showed that the area law holds at all $\beta$ for 3D $U(1)$ theory. This is still the only nontrivial case where the area law has been established at large $\beta$.

# Area law: Various other theorems

- Fröhlich (1979) showed that confinement holds in $SU(n)$ theory if it holds in the corresponding $\mathbb{Z}_n$ theory.
- Durhuus and Fröhlich (1980) showed that confinement in a $d$-dimensional pure lattice gauge theory holds if there is exponential decay of correlations in a $(d-1)$-dimensional nonlinear $\sigma$ model.
- Borgs and Seiler (1983) investigated confinement in lattice gauge theories at nonzero temperature, building on technology developed by Brydges and Federbush (1980).
- A toy model exhibiting a sharp transition from the confining to the deconfining regime was studied by Aizenman, Chayes, Chayes, Fröhlich and Russo (1983).

# Area law: Recent progress

- Area law at small $\beta$ for arbitrary loops (where area($\ell$) is the minimal surface area enclosed by $\ell$) was established in large $N$ limit of $SO(N)$ and $SU(N)$ theories by Chatterjee (2019), Jafarov (2016), and Chatterjee and Jafarov (2016).
- Chatterjee (2020a) computed the exact leading order behavior of Wilson loop expectations in 4D $\mathbb{Z}_2$ theory at large $\beta$.
- This result was extended to all 4D theories with finite Abelian gauge groups by Forsström, Lenells and Viklund (2020), and to all 4D theories with finite gauge groups by Cao (2020).

# Area law: Open questions

- The main open question about the area law is to prove that it holds in 4D $SU(3)$ theory at arbitrarily large values of $\beta$.
- In fact, proving it for any other 4D non-Abelian theory would be tremendously interesting too.
- It is not expected to hold for 4D Abelian theories at large $\beta$, nor even for 4D non-Abelian theories with finite gauge groups.

# Towards the main result: Assumptions

- Center symmetry: Assume that the center of $G$ is nontrivial, and there is an element $g_0$ in the center such that $\pi(g_0) = cI$ for some $c \neq 1$, where $I$ is the $m \times m$ identity matrix, $m$ being the dimension of $\pi$.
- Strong mass gap: Say that two edges are neighbors if they both belong to some common plaquette. Say that a measurable map $f : \Omega_N^\circ \to \mathbb{R}$ is a *local function* supported on an edge $e \in E_N$ if $f(\omega)$ depends only on the values of $\omega_u$ for $u$ that are neighbors of $e$. Given two local functions $f$ and $g$, let $\mathrm{dist}(f, g)$ denote the Euclidean distance between the midpoints of their supporting edges. Assume that there are positive constants $K_1$ and $K_2$ depending only on $G$, $\beta$ and $d$, and not on $N$ or the boundary condition $\delta$, such that for any local functions $f, g : \Omega_N^\circ \to [-1, 1]$,

$$|\langle fg\rangle - \langle f\rangle\langle g\rangle| \leq K_1 e^{-K_2 \,\mathrm{dist}(f,g)}.$$

# Main result

Theorem (C., 2020)

*Let $G$ be a compact connected subgroup of $U(n)$ for some $n$, and let $\pi$ be a finite-dimensional unitary representation of $G$. Take any $d \geq 2$ and $\beta \in \mathbb{R}$, and consider the lattice gauge theory on the cube $B_N$. Suppose that the center symmetry and strong mass gap assumptions are satisfied. Then there are positive constants $C_1$ and $C_2$ depending only on $G$, $\beta$, $\pi$ and $d$, such that the following holds. Take any $N \geq 2$, any boundary condition $\delta$ on $B_N$, and any rectangular loop $\ell$ contained in $B_{N'}$ for some $N' \leq N/2$. Then*

$$|\langle W_\ell \rangle| \leq C_1 e^{-C_2 \operatorname{area}(\ell)},$$

*where* $\operatorname{area}(\ell)$ *is the area enclosed by $\ell$. Moreover, there is a unique infinite volume Gibbs state, and the above bound holds for any rectangular loop $\ell$ if the expectation on the left is taken with respect to this Gibbs state.*

# Remarks

- As mentioned earlier, the strong mass gap assumption is stronger than usual mass gap, because mass gap means exponential decay of correlations under a specific boundary condition.
- The strong mass gap assumption is satisfied by any lattice gauge theory if $\beta$ is small enough. This allows our theorem to imply an old result of Osterwalder and Seiler (1978), but of course it says much more, since we have no requirement that $\beta$ be sufficiently small.
- It is not clear whether the strong mass gap assumption holds at all $\beta$ in 4D non-Abelian theories (as conjectured for usual mass gap), but it is not impossible. If this is proved, that will solve the confinement problem.
- This is the first rigorous result that throws light on the roles of mass gap and center symmetry in confinement, which are well-established in the theoretical physics discourse.

# Proof sketch: Step 1

- Let the first coordinate in $\mathbb{R}^d$ denote time.
- Let $\ell$ be a rectangular loop with side lengths $R$ and $T$, where the sides of length $T$ are parallel to the time axis.
- We wish to show that $|\langle W_\ell \rangle| \le C_1 e^{-C_2 RT}$.
- It is not very difficult to prove the perimeter law upper bound $|\langle W_\ell \rangle| \le C_1 e^{-C_2(R+T)}$.
- Given this, it is not hard to see that the area law can be established by proving the weaker bound

$$|\langle W_\ell \rangle| \le C_1 e^{C_2(R+T) - C_3 RT},$$

so this is what we will aim for.

# Proof sketch: Step 2

- Take one element from each of the matrices $\pi(\omega_e)$, $e \in \ell$, and let $f$ be the product of these elements.
- Such a variable will be called a component variable of $\ell$.
- Note that $W_\ell$ is a sum of $m^{2(R+T)}$ component variables, since

$$W_\ell = \mathrm{Tr}(\pi(\omega_{e_1}) \cdots \pi(\omega_{e_k})),$$

  where $e_1, \ldots, e_k$ are the edges of $\ell$.
- Thus, it suffices to prove that for any component variable $f$,

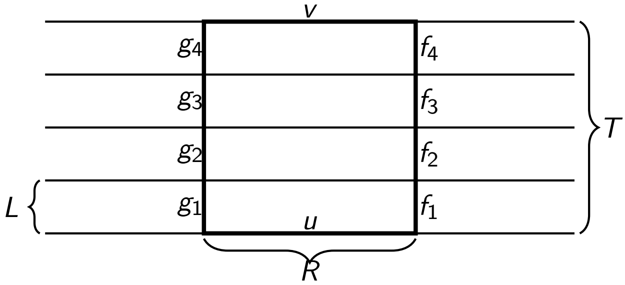
$$|\langle f \rangle| \le C_1 e^{C_2(R+T) - C_3 RT}.$$

# Proof sketch: Step 3

- Partition spacetime into slabs of thickness $L$ in the time direction, where $L$ will be chosen later, depending on $G$, $\beta$ and $d$.
- Let $r := T/L$. Then a component variable $f$ can be decomposed as

$$f = uvf_1 f_2 \cdots f_r g_1 g_2 \cdots g_r,$$

as depicted in the following diagram.



# Proof sketch: Step 4

- Let $\omega$ be a random configuration from the lattice gauge theory $\mu$.
- Let $\mu'$ be the conditional probability distribution given $\omega_e$ for all $e$ that belong to boundaries between slabs.
- Let $\langle f \rangle'$ denote the expected value of $f$ under $\mu'$, so that $\langle f \rangle = \langle \langle f \rangle' \rangle$.
- So it suffices to prove the area law for $\langle f \rangle'$ instead of $\langle f \rangle$.
- Under $\mu'$, the $\omega_e$'s inside one slab are independent of the $\omega_e$'s inside another slab, and the $\omega_e$'s on the boundaries are not random.
- Thus, we get the identity

$$\langle f \rangle' = uv \langle f_1 g_1 \rangle' \langle f_2 g_2 \rangle' \cdots \langle f_r g_r \rangle'.$$

- So, to show $|\langle f \rangle'| \le C_1 e^{C_2(R+T) - C_3 RT}$, it suffices to show that $|\langle f_i g_i \rangle'| \le C_4 e^{-C_5 R}$ for each $i$, since $r = T/L$ and $L$ is a constant.

# Proof sketch: Step 5

- Under $\mu'$, irrespective of the values of the $\omega_e$'s on the slab boundaries,
$$\langle f_i \rangle' = \langle g_i \rangle' = 0.$$
This holds because $G$ has a nontrivial center.
- Thus, we need to show that
$$|\langle f_i g_i \rangle' - \langle f_i \rangle' \langle g_i \rangle'| \leq C_1 e^{-C_2 R}.$$
- This will follow if we can show that correlations decay exponentially *within each slab*, for any given boundary condition on the slab.
- The key idea is to show that this holds if the thickness $L$ is chosen to be sufficiently large.

# Proof sketch: Step 6

- Take a large but finite slab

$$S = ([-N, N] \times [-M, M]^{d-1}) \cap \mathbb{Z}^d,$$

where $M \gg N$.
- The set of all boundary edges of $S$ that do not belong to either the top face or the bottom face will be called the spatial boundary of $S$.
- Consider lattice gauge theory in $S$ with some given boundary condition.
- We will show that the influence of the spatial boundary near the center of $S$ is exponentially small in $M$ when $N$ is fixed and $M \to \infty$, provided that $N$ is sufficiently large, depending on $G$, $\beta$ and $d$.
- This suffices to show exponential decay of correlations in a sufficiently thick slab.

# Proof sketch: Step 7

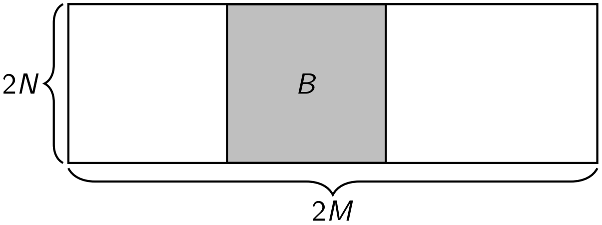
- Take two boundary conditions on $S$ that differ only on the spatial boundary.
- Let $\mu$ and $\mu'$ be the probability measures defined by these two boundary conditions.
- A coupling of $\mu$ and $\mu'$ is a pair of random configurations $(\omega, \omega')$ such that $\omega \sim \mu$ and $\omega' \sim \mu'$.
- Our goal would be achieved if we can construct a coupling such that for any edge $e$ near the center of $S$, the chance of $\omega_e \neq \omega'_e$ is exponentially small in $M$.
- Such a coupling is constructed in three steps.

# Construction of the coupling: Step 1

- First, we construct a coupling with similar properties in the cube $B_N$ instead of the slab $S$, using the strong mass gap assumption and the coupling characterization of total variation distance between probability measures.
- The connectedness of the gauge group is used in this step, to pass smoothly from one boundary condition to another and bounding the rate of change using two-point correlations.

# Construction of the coupling: Step 2

- Given any coupling $(\omega, \omega')$ on $S$, we update the coupling to get a better coupling as follows.
- Choose a copy $B$ of $B_N$ inside $S$ uniformly at random. Fixing $\omega_e$ and $\omega'_e$ outside the interior of $B$, replace the values in the interior by a coupled pair generated using the mechanism from the first step.




- The resulting pair of configurations is again a coupling of $\mu$ and $\mu'$, and is an 'improvement' of the original coupling. The 'amount of improvement' is quantified through an inequality.

# Construction of the coupling: Step 3

- The updating is repeated an infinite number of times to get a vastly improved coupling as a subsequential limit.
- Applying the inequality from the previous step, one can use this coupling to prove exponential decay of correlations in a slab if the thickness of the slab is large enough.
- The condition that the thickness has to be large is required to ensure that a certain parameter in the inequality is strictly less than one, which leads to the exponential decay.

# Preprint

Chatterjee, S. (2020). A probabilistic mechanism for quark confinement. *Preprint.* Available at https://arxiv.org/abs/2006.16229.