Chaos, concentration, and multiple valleys

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- The minimum energy path (of length n) is called the ground state of the system.

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- What we have is small change in all coordinates.
- The two notions of perturbation seem to have the same macroscopic effects.

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- No rigorous results.
- Similar conjectures for many other models.

Chaos in directed polymers: A rigorous result

Theorem

Fix n, and let $t_0 = (\log n)^{-1/2}$. *Then for any* $t \geq t_0$,

$$\mathbb{E}|\hat{p} \cap \hat{p}^t| \leq \frac{Cn}{\sqrt{\log n}}$$

where C is a universal constant.

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For $t \geq n^{-1/6}$,

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- Here the state space is S^{n-1} and the energy of $x \in S^{n-1}$ is $-x^T Ax$. The first eigenvector is the ground state.

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- Important open question in spin glasses: If X_{ij} are i.i.d. Gaussian, is the field $Z(\sigma) = \sum X_{ij} \sigma_i \sigma_j$ chaotic?

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- Incidentally, Johansson (2000) showed via a miraculous connection with random matrices, that the variance is of order $n^{2/3}$ when ω_v are i.i.d. Geometric random variables.

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Theorem

There are constants $\ell_n \to \infty$, $\gamma_n \to 0$, $\epsilon_n \to 0$, and $\delta_n \to 0$ such that with probability at least $1 - \gamma_n$, there is a set A of paths (of length n) satisfying

1. $|A| \geq \ell_n$,
2. $|p \cap p'| \leq \epsilon_n n$ *for all* $p, p' \in A$, $p \neq p'$, *and*
3. $\omega(p) \geq (1 - \delta_n)\omega(\hat{p})$ *for all* $p \in A$.

(Exact quantitative version is available.)

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- Instead of the optimizing path we consider the optimizing index $I^t := \text{argmax}_{i \in S} X_i^t$.

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- Precise version: We call a sequence of Gaussian fields $\mathbf{X}_n$ chaotic if $\exists\ t_n \to 0$ such that

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- By being more demanding about the definition of near-maximality, one can define strong multiple peaks.

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Moreover, under the positivity assumption that $R(i,j) \geq 0$ for each i,j,

$$\textit{Strong MP} \underset{\not\Longleftarrow}{\Longrightarrow} \textit{Superconcentration} \iff \textit{Chaos} \underset{\not\Longleftarrow}{\Longrightarrow} \textit{Multiple Peaks}.$$

Proof of superconcentration $\iff$ chaos

- Gaussian integration by parts: For any smooth g,

$$\mathbb{E}(X_i g(\mathbf{X})) = \sum_{j\in S} R(i,j)\mathbb{E}\left(\frac{\partial g}{\partial x_i}(\mathbf{X})\right).$$

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- Recall: $\mathbf{X}^t = e^{-t}\mathbf{X} + \sqrt{1-e^{-2t}}\mathbf{X}'$.
- Let $f(\mathbf{X})$ be a smooth approximation of $\max_{i\in S} X_i$. Then

$$\mathrm{Var}(f(\mathbf{X})) = \mathbb{E}\big[f(\mathbf{X})(f(\mathbf{X}) - f(\mathbf{X}'))\big] = \mathbb{E}\left(-\int_0^\infty f(\mathbf{X})\frac{d}{dt}f(\mathbf{X}^t)dt\right)$$

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- The integral approaches $\int_0^\infty e^{-t}\mathbb{E}(R(I^0, I^t))dt$ as $f(\mathbf{X}) \to \max X_i$.

Proof contd.

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- From this and (1) we can deduce that
$$\mathbb{E}(R(I^0, I^t)) \le \frac{\mathrm{Var}(\max X_i)}{1 - e^{-t}} \text{ and}$$
$$\mathrm{Var}(\max X_i) \le (1 - e^{-t}) \max \mathrm{Var}(X_i) + \mathbb{E}(R(I^0, I^t))e^{-t}.$$

This proves the equivalence of chaos and superconcentration.

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- Iterating this procedure, we can find many such points.
- If the motion of I^t does not have big jumps, then we have bridges between distant near-maxima. Can prove for eigenvectors.

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- Introduced by Sherrington & Kirkpatrick in 1975. Source of a large body of groundbreaking physics and deep mathematics, culminating in Talagrand's proof of the Parisi formula in 2006. Still, many open questions.

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- No existing method can show that the fluctuation exponent is $< 1/2$.
- We show that, under a certain domination condition on the sequence $\mathbf{c}$, the fluctuation exponent is $\le 3/8$.
- The proof uses a new technique, that may have the potential to be developed as an alternative to hypercontractivity for variance bounds.

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- We know that $\sum_{p=2}^{\infty} c_p = 1$. So $\lim_{p\to\infty} c_p = 0$. What we want is that c_p does not decay to zero 'too fast', in the following sense.

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where c_p^* are the coefficients in the power series expansion of the function

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- $c_p^* \geq 0$ for all p and $\sum_{p=2}^{\infty} c_p^* = 1$.
- Unfortunately, does not cover the classical p-spin models.

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- Chaos is one of the two most important conjectures about mean-field spin glasses. No rigorous proof in any nontrivial model.
- We have a proof of chaos with precise bounds under the domination condition.
- Similar remark for multiple valleys: no rigorous results in the literature; we can prove under the domination condition.

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Besides the stated examples, we have proved chaos, multiple valleys, and anomalous fluctuations in a number of other models:

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- Branching random walk with Gaussian increments.

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- Applications to polymers, spin glasses, fitness models, the Gaussian free field, eigenvectors of random matrices, Gaussian fields on Euclidean spaces, branching random walks.
- Not discussed in the talk: (a) Modifications of the hypercontractive method, and (b) an alternative to hypercontractivity that can reduce fluctuation exponents to $< 1/2$ rather than just logarithmic improvements.