The universal relation between exponents in first-passage percolation

Sourav Chatterjee

(Courant Institute, NYU)

First-passage percolation

- Let $E(\mathbb{Z}^d)$ denote the set of edges in the integer lattice $\mathbb{Z}^d$.
- Each edge e has a 'weight' or 'passage time' attached to it, denoted by t_e.
- Ordinary first-passage percolation assumes that t_e are i.i.d. non-negative random variables. Introduced by Hammersley & Welsh (1960).
- The total passage time, or total weight, of a path P is simply the sum of the weights of the edges in P.
- The first-passage time $T(x, y)$ from a point x to a point y is the minimum total passage time among all lattice paths from x to y.
- If the edge-weights are continuous random variables, then with probability one there is a unique weight minimizing path (geodesic) between any x and y that we call $G(x, y)$.
- Let $D(x, y)$ denote the maximum deviation of this path from the straight line segment joining x and y.

$G(x, y)$ and $D(x, y)$

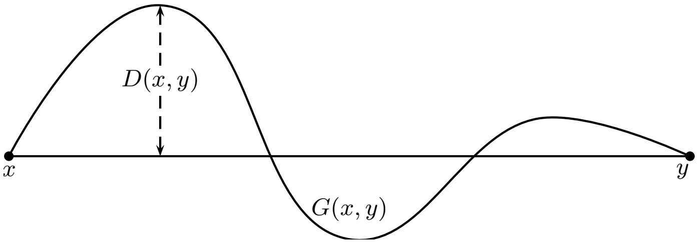


Figure: The geodesic $G(x, y)$ and the deviation $D(x, y)$.

Exponents

- The fluctuation exponent χ is a number that quantifies the order of fluctuations of the first-passage time $T(x, y)$.
- Roughly speaking, for any x, y, the typical value of $T(x, y) - \mathbb{E}T(x, y)$ is of the order $|x - y|^\chi$.
- One possible definition: χ is the smallest number such that

$$\left\{ \frac{T(x, y) - \mathbb{E}T(x, y)}{|x - y|^\chi} \right\}_{x \neq y} \quad \text{is a tight family.}$$

- The wandering exponent ξ quantifies the magnitude of $D(x, y)$. Again, roughly speaking, for any x, y, the typical value of $D(x, y)$ is of the order $|x - y|^\xi$.

The conjectured relation between χ and ξ

- Physicists say that irrespective of the dimension, χ and ξ must satisfy the scaling identity $\chi = 2\xi - 1$.
- Conjectured in numerous physics papers: Huse and Henley (1985), Kardar, Parisi and Zhang (1986), Kardar and Zhang (1987), Krug (1987), Krug and Meakin (1989), Krug and Spohn (1991), Meakin et. al. (1986), Medina et. al. (1989), Wolf and Kertész (1987), etc. Sometimes called KPZ relation.
- Rigorous literature gives support in one direction only: Newman and Piza (1995) showed that $\chi' \geq 2\xi - 1$, where χ' is a related exponent that may be equal to χ. Earlier, Wehr and Aizenman (1990) proved $\chi \geq (1 - (d-1)\xi)/2$.
- Wüthrich (1998) proved the opposite inequality for Sznitman's model of Brownian motion in a truncated Poissonian potential. No results for first-passage percolation.
- In this talk, I will prove the relation $\chi = 2\xi - 1$ assuming that χ and ξ exist in a certain sense.

Defining χ 'from above'

- Recall that a family of random variables $\{X_n\}$ is said to be exponentially tight if there exists $\alpha > 0$ such that $\mathbb{E}(e^{\alpha|X_n|})$ is uniformly bounded as n varies.
- Kesten (1993) proved that $\chi \le 1/2$ in the sense that the family

$$\left\{ \frac{T(x,y) - \mathbb{E}T(x,y)}{|x-y|^{1/2}} \right\}_{x \neq y}$$

  is exponentially tight.
- Inspired by Kesten's result, one may define an exponent χ_a 'from above' to be the infimum of all δ such that the family

$$\left\{ \frac{T(x,y) - \mathbb{E}T(x,y)}{|x-y|^{\delta}} \right\}_{x \neq y}$$

  is exponentially tight.

Defining χ from below

- It is easy to show under mild assumptions that there is a $C > 0$ such that for all $x \neq y$

$$\mathrm{Var}(T(x,y)) \geq C.$$

- As before, this inspires the definition of an exponent χ_b 'from below' as the supremum of all δ such that for some $C > 0$, for all $x \neq y$,

$$\mathrm{Var}(T(x,y)) \geq C|x-y|^{2\delta}.$$

- It is easy to show that $0 \leq \chi_b \leq \chi_a \leq 1/2$. When $\chi_a = \chi_b$, we will say that the fluctuation exponent χ exists and equals this number.
- The wandering exponent ξ may be defined in a similar manner, by first defining ξ_a and ξ_b, and then saying that ξ exists if $\xi_a = \xi_b$.

BKS distributions

- It follows from Kesten's result that

$$\mathrm{Var}(T(x,y)) \le C|x-y|.$$

- For binary edge weights, this was improved by Benjamini, Kalai and Schramm (2003), using a hypercontractive method of Talagrand:

$$\mathrm{Var}(T(x,y)) \le C\frac{|x-y|}{\log|x-y|}.$$

- We will say that the edge-weight distribution belongs to the BKS class if the above improved bound holds. Thus, two-point distributions belong to the BKS class.
- Benaïm and Rossignol (2008) proved that a large class of distributions, including the exponential, beta, gamma and uniform distributions, belong to the BKS class.

The main result

Theorem (C., 2011)

Consider first-passage percolation in any dimension ≥ 2. Suppose that the edge-weight distribution is continuous and belongs to the BKS class, and that the exponents χ and ξ exist in the sense defined before. Then $\chi = 2\xi - 1$.

Remark: Recently, Auffinger and Damron have made an important improvement to the proof that removes the BKS assumption. Will say more about it later.

The functions g and h

- Let $h(x) := \mathbb{E}(T(0,x))$.
- The function h is subadditive. Therefore the limit

$$g(x) := \lim_{n\to\infty} \frac{h(nx)}{n}$$

  exists for all $x \in \mathbb{Z}^d$.
- Can be extended to all $x \in \mathbb{Q}^d$ by taking $n \to \infty$ through a subsequence.
- Can be extended to all $x \in \mathbb{R}^d$ by uniform continuity.
- The function g is a norm on $\mathbb{R}^d$.

Approximation of h by g

- g is a norm, and hence much more well-behaved than h.
- If $|x|$ is large, $g(x)$ is supposed to be a good approximation of $h(x)$.
- Alexander's method: Use the order of fluctuations of passage times to infer bounds on $|h(x) - g(x)|$.
- For any $\varepsilon > 0$, there exists C such that for all $x \neq 0$,

$$g(x) \leq h(x) \leq g(x) + C|x|^{\chi+\varepsilon}.$$

- In the proof of the main result, the above approximation will allow us to replace the expected passage time $h(x)$ by the norm $g(x)$.

Curvature lemma

- There is a unit vector x_0 and a hyperplane H_0 perpendicular to x_0 such that for some $C > 0$, for all $z \in H_0$,

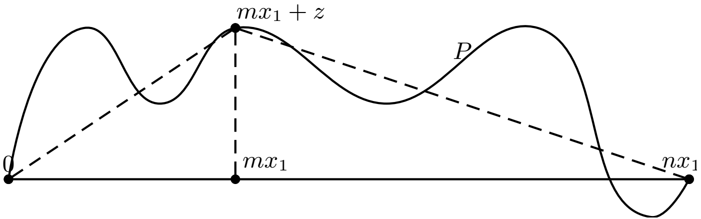
$$|g(x_0 + z) - g(x_0)| \leq C|z|^2.$$

- There is a unit vector x_1 and a hyperplane H_1 perpendicular to x_1 such that for some $C > 0$, for all $z \in H_1$, $|z| \leq 1$,

$$g(x_1 + z) \geq g(x_1) + C|z|^2.$$

- In the direction x_0, the unit sphere of the norm g is 'at most as curved as an Euclidean sphere' and in the direction x_1, it is 'at least as curved as an Euclidean sphere'.

Idea of the proof of $\chi \geq 2\xi - 1$



- By the definition of the direction of curvature x_1,

$$\begin{aligned} &\text{Expected passage time of the path } P \\ &\geq g(mx_1 + z) + g(nx_1 - (mx_1 + z)) + O(n^{\chi+\varepsilon}) \\ &\geq \mathbb{E}(T(0, nx_1)) + C|z|^2/n + O(n^{\chi+\varepsilon}). \end{aligned}$$

- Suppose $|z| = n^{\xi}$. Then $|z|^2/n = n^{2\xi-1}$.
- Fluctuations of $T(0, nx_1)$ are of order n^{χ}. Thus, if $2\xi - 1 > \chi$, then P cannot be a geodesic from 0 to nx_1.

Proving $\chi \leq 2\xi - 1$ when $\chi > 0$

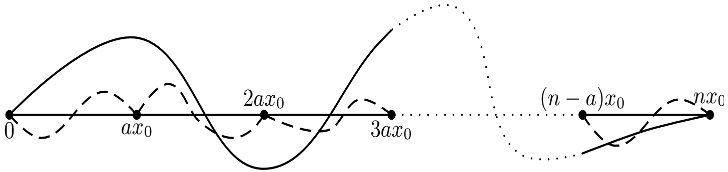


Figure: Solid curve is $G(0, nx_0)$. Dashed curves are $G(iax_0, (i+1)ax_0)$

- Recall direction of curvature x_0. Let $a = n^\beta$, $\beta < 1$. Let $m = n/a = n^{1-\beta}$.
- Under the condition $\chi > 2\xi - 1$, we will show that there is a $\beta < 1$ such that

$$T(0, nx_0) = \sum_{i=0}^{m-1} T(iax_0, (i+1)ax_0) + o(n^\chi). \qquad (\star)$$

This will lead to a contradiction.

Getting a contradiction under ($\star$)

- Let $f(n) := \mathrm{Var}\, T(0, nx_0)$. Then by the BKS criterion, $f(n) \le Cn/\log n$.
- Under ($\star$), by the FKG-Harris inequality,

$$\begin{aligned} f(n) = \mathrm{Var}\, T(0, nx_0) &\ge m\mathrm{Var}\, T(0, ax_0) + o(n^{2\chi}) \\ &= n^{1-\beta} f(n^\beta) + o(n^{2\chi}). \end{aligned}$$

- Consequently,

$$\liminf_{n\to\infty} \frac{f(n)}{n^{1-\beta} f(n^\beta)} \ge 1. \qquad (\dagger)$$

- Choose $n_0 > 1$ and define $n_{i+1} = n_i^{1/\beta}$ for each i.
- Let $v(n) := f(n)/n$. Then $v(n_i) \le C/\log n_i \le C\beta^i$.
- But by ($\dagger$), $\liminf v(n_{i+1})/v(n_i) \ge 1$, and so for all i large enough, $v(n_{i+1}) \ge \beta^{1/2} v(n_i)$.
- In particular, $v(n_i) \ge const.\beta^{i/2}$.
- Since $\beta < 1$, this gives a contradiction for i large.

Proving ($\star$): Small blocks and big blocks

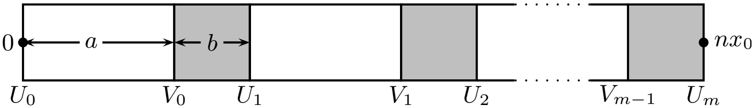


Figure: Cylinder of width n^{ξ} around the line joining 0 and nx_0

- Let $a = n^{\beta}$ and $b = n^{\beta'}$, where $\beta' < \beta < 1$.
- Consider a cylinder of width n^{ξ} around the line joining 0 and nx_0.
- Partition the cylinder into alternating big and small cylinders of widths a and b respectively.
- Call the boundary walls of these cylinders $U_0, V_0, U_1, V_1, \ldots, V_{m-1}, U_m$, where m is roughly $n^{1-\beta}$.

Approximating by cylinders

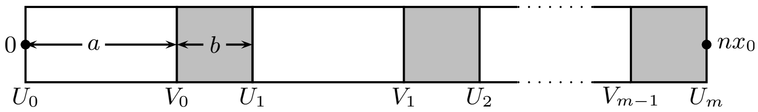


Figure: Cylinder of width n^ξ around the line joining 0 and nx_0

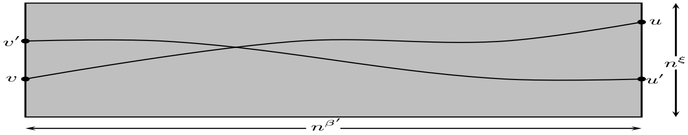
- Key step: (will skip the proof)

$$\left| T(U_0, U_m) - \sum_{i=0}^{m-1} (T(U_i, V_i) + T(V_i, U_{i+1})) \right| \le \sum_{i=0}^{m-1} M_i,$$

where $M_i := \max_{v,v' \in V_i,\ u,u' \in U_{i+1}} |T(v,u) - T(v',u')|$.

- Note that the errors M_i come only from the small blocks.

Proving ($\star$): Estimating the error



- By curvature estimate in direction x_0,

$$|\mathbb{E}T(v,u) - \mathbb{E}T(v',u')| \leq C(n^{\xi})^2/n^{\beta'} = Cn^{2\xi-\beta'}.$$

- Fluctuations of $T(v,u)$ are of order $n^{\beta'\chi}$.
- If $2\xi - 1 < \chi$, then we can choose β' so close to 1 that $2\xi - \beta' < \beta'\chi$. That is, fluctuations dominate while estimating M_i. Consequently, M_i is of order $n^{\beta'\chi}$.
- Thus, total error $= n^{1-\beta+\beta'\chi}$. Since $\beta' < \beta$ and $\chi > 0$, this allows us to choose β', β such that the exponent is $< \chi$. This proves ($\star$).
- The above bound has been cleverly used by Auffinger and Damron to get a simpler proof of $\chi \leq 2\xi - 1$ when $\chi > 0$.

The case $\chi = 0$

- The proof of $\chi \leq 2\xi - 1$ when $\chi = 0$ is quite different than the case $\chi > 0$. I will spare you the details.

A digression: superconcentration and chaos

- It is conjectured that $\chi = 1/3$ and $\xi = 2/3$ is 2D. It is also believed that χ may be 0 in sufficiently high dimension.
- As stated before, Kesten showed that $\chi \le 1/2$. However, it is not even known whether $\chi < 1/2$ in any dimension.
- The phenomenon that $\chi < 1/2$ is sometimes called 'sublinear variance'. I call it 'superconcentration'. Manifests in many other models.
- In the related model of last-passage percolation with Gaussian weights, I proved in 2008 that $\chi = \frac{1}{2}\tau$, where τ is an exponent such that $|P \cap P'|$ is of order n^τ, where P is the optimal path and P' is the optimal path in a slightly perturbed environment.
- In other words, $\chi < 1/2$ if and only if $\tau < 1$. The phenomenon $\tau < 1$ is sometimes called 'chaos' in such disordered systems.
- Main result of my 2008 paper: superconcentration is equivalent to chaos.

Preprints on arXiv

- S. Chatterjee (2008): Chaos, concentration, and multiple valleys.
- S. Chatterjee (2011): The universal relation between scaling exponents in first-passage percolation.
- A. Auffinger and M. Damron (2011): A simplified proof of the relation between scaling exponents in first-passage percolation.