

The $1/N$ expansion for lattice gauge theories

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Yang–Mills theories

- ▶ Maxwell's equations are a set of four equations that describe the behavior of an electromagnetic field.
- ▶ Hermann Weyl showed that these four equations are actually the Euler–Lagrange equations for an elegant minimization problem.
- ▶ In modern parlance, Maxwell's equations minimize the Yang–Mills functional for the gauge group $U(1)$.
- ▶ Physicists later realized that two of the other three fundamental forces of nature — the weak force and the strong force — can also be modeled by similar equations: one needs to simply change the group $U(1)$ to some other group ($SU(3)$ for the strong force, $SU(2)$ for the weak force). This led to the formulation of the **Standard Model**.
- ▶ Equations obtained by minimizing Yang–Mills functionals over the space of connections of a principal bundle.

Connection form

- ▶ A quantum YM theory starts with a compact Lie group, called the **gauge group**.
- ▶ In this talk, the gauge group is $SO(N)$.
- ▶ Recall that the Lie algebra $\mathfrak{so}(N)$ of the Lie group $SO(N)$ is the set of all $N \times N$ skew-symmetric matrices.
- ▶ An **$SO(N)$ connection form** on \mathbb{R}^d is a smooth map from \mathbb{R}^d into $\mathfrak{so}(N)^d$.
- ▶ If A is a $SO(N)$ connection form, its value $A(x)$ at a point x is a d -tuple $(A_1(x), \dots, A_d(x))$ of skew-symmetric matrices.
- ▶ In the language of differential forms, $A = \sum_{j=1}^d A_j dx_j$.

- ▶ The curvature form F of a connection form A is the 2-form $F = dA + A \wedge A$.
- ▶ Explicitly, $F(x)$ is a $d \times d$ array of skew-symmetric matrices of order N , whose $(j, k)^{\text{th}}$ entry is the matrix

$$F_{jk}(x) = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j(x), A_k(x)].$$

Yang–Mills action

- ▶ Let \mathcal{A} be the space of all $SO(N)$ connection forms on \mathbb{R}^d .
- ▶ The Yang–Mills action on \mathcal{A} is the function

$$S_{\text{YM}}(A) := - \int_{\mathbb{R}^d} \text{Tr}(F \wedge *F),$$

where F is the curvature form of A and $*$ denotes the Hodge $*$ -operator.

- ▶ Explicitly,

$$S_{\text{YM}}(A) = - \int_{\mathbb{R}^d} \sum_{j,k=1}^d \text{Tr}(F_{jk}(x)^2) dx.$$

Physics definition of Euclidean Yang–Mills theory

$SO(N)$ quantum Yang–Mills theory in d -dimensional Euclidean spacetime is formally described as the probability measure

$$d\mu(A) = \frac{1}{Z} \exp\left(-\frac{1}{4g^2} S_{\text{YM}}(A)\right) \mathcal{D}A,$$

where A belongs to the space \mathcal{A} of all $SO(N)$ connection forms, S_{YM} is the Yang–Mills functional,

$$\mathcal{D}A = \prod_{j=1}^d \prod_{x \in \mathbb{R}^d} d(A_j(x))$$

is “infinite dimensional Lebesgue measure” on \mathcal{A} , g a positive coupling constant, and Z is the normalizing constant (partition function) that makes this a probability measure.

The Yang–Mills existence problem

- ▶ The physics definition of Euclidean Yang–Mills theory is not mathematically valid, due to the non-existence of infinite dimensional Lebesgue measure on \mathcal{A} .
- ▶ The Yang–Mills existence problem has two parts: First, give a rigorous mathematical definition of Euclidean YM theory. Second, extend the theory to Minkowski spacetime by Wick rotation.
- ▶ The second part can be deduced from the first by standard tools from constructive quantum field theory if the Euclidean theory can be shown to satisfy certain properties (Wightman axioms or Osterwalder-Schrader axioms).

- ▶ In 1974, Kenneth Wilson introduced a discrete approximation of Euclidean YM theory, that is now known as lattice gauge theory.
- ▶ Lattice gauge theories are well-defined probability measures on subsets of \mathbb{Z}^d .

Wilson's discretization

- ▶ Let $A = \sum_{j=1}^d A_j dx_j$ be an $SO(N)$ connection form on \mathbb{R}^d .
- ▶ Discretize \mathbb{R}^d as $\epsilon\mathbb{Z}^d$.
- ▶ Let e_1, \dots, e_d be the standard basis.
- ▶ For any $x \in \epsilon\mathbb{Z}^d$ and $1 \leq j \leq d$, let $U(x, x + \epsilon e_j) := e^{\epsilon A_j(x)}$.
- ▶ Let $U(y, x) := U(x, y)^{-1}$ for any edge (x, y) of $\epsilon\mathbb{Z}^d$.
- ▶ Take any x and $1 \leq j < k \leq d$. Let x_1, x_2, x_3, x_4 be the four vertices $x, x + \epsilon e_j, x + \epsilon e_j + \epsilon e_k$ and $x + \epsilon e_k$.
- ▶ Define $U(x, j, k)$
$$:= U(x_1, x_2)U(x_2, x_3)U(x_3, x_4)U(x_4, x_1)$$
$$= e^{\epsilon A_j(x_1)} e^{\epsilon A_k(x_2)} e^{-\epsilon A_j(x_4)} e^{-\epsilon A_k(x_1)}.$$

- ▶ Observe that

$$A_k(x_2) = A_k(x + \epsilon e_j) \approx A_k(x) + \epsilon \frac{\partial A_k}{\partial x_j},$$

$$A_j(x_4) = A_j(x + \epsilon e_k) \approx A_j(x) + \epsilon \frac{\partial A_j}{\partial x_k}.$$

- ▶ Recall Baker–Campbell–Hausdorff formula

$$e^B e^C = e^{B+C+\frac{1}{2}[B,C]+\dots}.$$

- ▶ Using the last two displays, one can show that

$$\begin{aligned} & \text{Tr}(I - U(x, j, k)) \\ & \approx -\frac{\epsilon^4}{2} \text{Tr} \left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j(x), A_k(x)] \right)^2 \\ & = -\frac{\epsilon^4}{2} \text{Tr}(F_{jk}(x)^2). \end{aligned}$$

Wilson's definition of $SO(N)$ lattice gauge theory

- ▶ Let Λ be a finite subset of \mathbb{Z}^d . Take any $\epsilon > 0$.
- ▶ On each edge (x, y) of $\epsilon\Lambda$, attach an $SO(N)$ matrix $U(x, y)$ with the constraint that $U(y, x) = U(x, y)^{-1}$.
- ▶ Any such assignment of matrices to edges will be called a configuration.
- ▶ Define $U(x, j, k)$ as in the previous slide, and let $\mu_{\Lambda, \epsilon, g}$ be the probability measure on the set of all configurations that has density

$$\frac{1}{Z(\Lambda, \epsilon, g)} \exp\left(-\frac{1}{g^2 \epsilon^{4-d}} \sum_{x, j, k} \text{Tr}(I - U(x, j, k))\right)$$

with respect to the product Haar measure. Here $Z(\Lambda, \epsilon, g)$ is the partition function (normalizing constant).

- ▶ The probability measure $\mu_{\Lambda, \epsilon, g}$ is called **$SO(N)$ lattice gauge theory on $\epsilon\Lambda$ with coupling strength g** .

Wilson loops

- ▶ In this talk, we will not be worried about continuum limits. We will simply set $\epsilon = 1$.
- ▶ The coefficient $1/g^2$ will be reparametrized as $N\beta$. This is known as the 't Hooft scaling.
- ▶ Lastly, we will replace the finite set Λ by the whole of \mathbb{Z}^d by taking an infinite volume limit.
- ▶ The main objects of study in lattice gauge theories are **Wilson loop variables**. Introduced by Wilson to understand **quark confinement**.
- ▶ A Wilson loop ℓ is simply a closed path in the lattice \mathbb{Z}^d .
- ▶ One can look at ℓ as a sequence of edges e_1, e_2, \dots, e_n .
- ▶ The **Wilson loop variable** W_ℓ is defined as

$$W_\ell := \text{Tr}(U(e_1)U(e_2)\cdots U(e_n)).$$

- ▶ **Key question:** Compute $\langle W_\ell \rangle$, where $\langle \cdot \rangle$ denotes expectation under the lattice gauge theory.

In case you are totally lost:

- ▶ Just keep in mind that $SO(N)$ lattice gauge theory on \mathbb{Z}^d involves a single parameter β , known as the inverse coupling strength.
- ▶ Given β , the theory describes a certain probability measure on the space of all configurations of $SO(N)$ matrices attached to the edges of \mathbb{Z}^d .
- ▶ If ℓ is a loop in \mathbb{Z}^d , the Wilson loop variable W_ℓ is the trace of the product of the matrices attached to the edges of ℓ .
- ▶ The Wilson loop expectation $\langle W_\ell \rangle$ is the expected value of W_ℓ in the above theory.

The $1/N$ expansion

- ▶ The $1/N$ expansion for Wilson loop expectations is a way of expressing $\langle W_\ell \rangle$ as a **formal power series**:

$$\frac{\langle W_\ell \rangle}{N} = f_0(\ell) + \frac{f_1(\ell)}{N} + \frac{f_2(\ell)}{N^2} + \dots,$$

where the functions f_0, f_1, \dots may depend on the coupling parameter β but not on N .

- ▶ 't Hooft's idea: Express the left side in terms of matrix integrals and use planar diagrams to express f_k as a sum over surfaces of genus k .

Planar diagram theory

- ▶ For matrix integrals involving finitely many matrices, rigorous mathematical theory developed over many years by many authors: Ercolani, McLaughlin, Eynard, Orantin, Zinn-Justin, ... and especially **Guionnet and collaborators** [Collins, Jones, Maïda, Maurel-Segala, Novak, Shlyakhtenko, Zeitouni, ...]
- ▶ Lattice gauge theories, however, involve infinitely many matrices.
- ▶ This introduces certain difficulties that necessitate the development of a new approach. In particular, I had to replace the sum over planar diagrams by a sum over trajectories in a lattice string theory.

Theorem (C. & Jafarov, 2016)

Consider $SO(N)$ lattice gauge theory on \mathbb{Z}^d with coupling parameter β . Take any $k \geq 0$. There exists $\beta_0(k, d) > 0$ such that if $|\beta| \leq \beta_0(k, d)$, then for any loop ℓ ,

$$\frac{\langle W_\ell \rangle}{N} = f_0(\ell) + \frac{f_1(\ell)}{N} + \dots + \frac{f_k(\ell)}{N^k} + o\left(\frac{1}{N^k}\right)$$

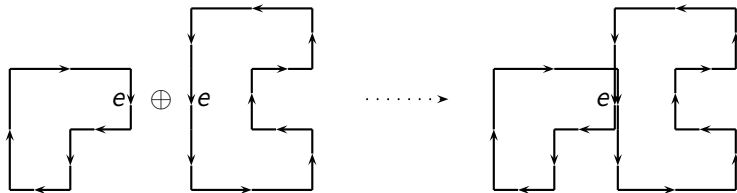
as $N \rightarrow \infty$, where $f_0(\ell), f_1(\ell), \dots$ are expressible as absolutely convergent sums over trajectories in a certain 'lattice string theory' on \mathbb{Z}^d (to be described in subsequent slides).

- ▶ The case $k = 0$ was proved in an earlier paper (C., 2015).
- ▶ The fact that there is often a 'duality' between gauge theories and string theories is well-known to physicists and is currently a very active area of research. The above theorem is probably the first rigorous result on this topic. (See **AdS-CFT duality**.)

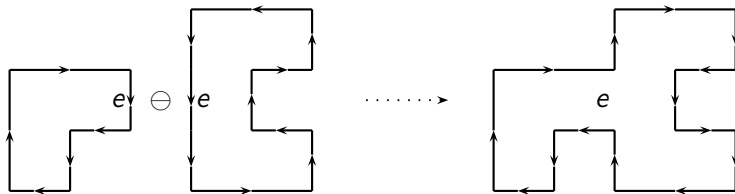
Lattice string theory

- ▶ In our definition, a string is a finite collection of loops in \mathbb{Z}^d .
- ▶ The string evolves over time.
- ▶ At each time step, a component loop may become slightly deformed, or two loops may merge, or a loop may split into two, or a loop may twist at a bottleneck, or nothing may happen.
- ▶ Each operation has two subtypes. Total of eight possible operations (nine, if we count doing nothing as an operation).
- ▶ These operations are described pictorially in the following slides.

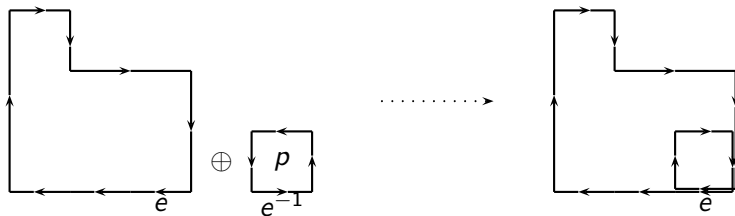
Positive merger



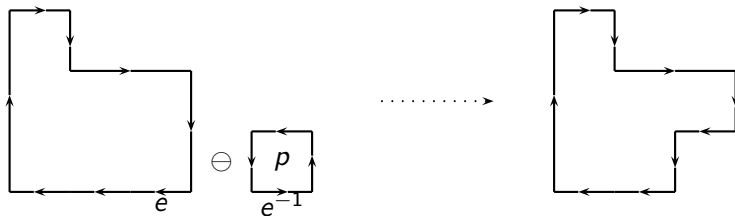
Negative merger



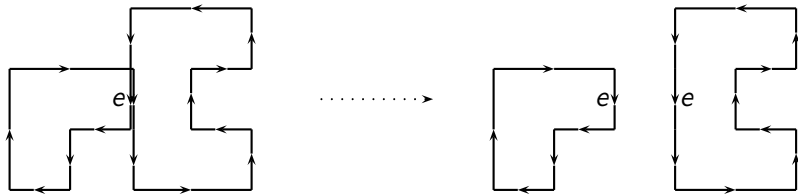
Positive deformation



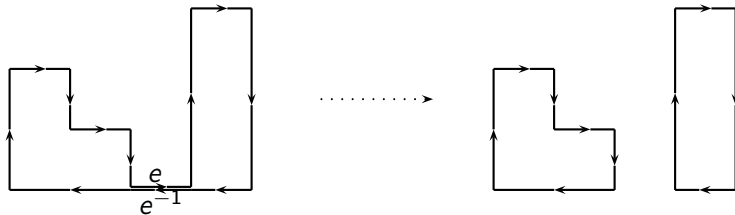
Negative deformation



Positive splitting



Negative splitting



Operations on lattice strings

- ▶ Recall: a string s is a finite collection of loops.
- ▶ After performing one of the eight operations, we get a new string s' , which may contain a different number of loops.
- ▶ Let

$$\mathbb{D}^+(s) := \{s' : s' \text{ is a positive deformation of } s\},$$

$$\mathbb{D}^-(s) := \{s' : s' \text{ is a negative deformation of } s\},$$

$$\mathbb{S}^+(s) := \{s' : s' \text{ is a positive splitting of } s\},$$

$$\mathbb{S}^-(s) := \{s' : s' \text{ is a negative splitting of } s\},$$

$$\mathbb{M}^+(s) := \{s' : s' \text{ is a positive merger of } s\},$$

$$\mathbb{M}^-(s) := \{s' : s' \text{ is a negative merger of } s\},$$

$$\mathbb{T}^+(s) := \{s' : s' \text{ is a positive twisting of } s\},$$

$$\mathbb{T}^-(s) := \{s' : s' \text{ is a negative twisting of } s\}.$$

Weight of a transition

- ▶ Let s be a string and $|s|$ be the total number of edges in s .
- ▶ If s evolves into s' after an operation, define the weight of the transition from s to s' at inverse coupling strength β as

$$w_{\beta}(s, s') := \begin{cases} 1 & \text{if } s' = s, \\ -1/|s| & \text{if } s' \in \mathbb{T}^{+}(s) \cup \mathbb{S}^{+}(s) \cup \mathbb{M}^{+}(s), \\ 1/|s| & \text{if } s' \in \mathbb{T}^{-}(s) \cup \mathbb{S}^{-}(s) \cup \mathbb{M}^{-}(s), \\ -\beta/|s| & \text{if } s' \in \mathbb{D}^{+}(s), \\ \beta/|s| & \text{if } s' \in \mathbb{D}^{-}(s). \end{cases}$$

- ▶ If $X = (s_0, s_1, \dots, s_n)$ is a **trajectory** of strings (that is, each s_{i+1} is obtained from s_i by an operation), define the weight of X as

$$w_{\beta}(X) = w_{\beta}(s_0, s_1)w_{\beta}(s_1, s_2) \cdots w_{\beta}(s_{n-1}, s_n).$$

- ▶ A trajectory as above is called **vanishing** if s_n is empty and s_{n-1} is not.

Genus of a string trajectory

- ▶ Suppose that a vanishing string trajectory has a twistings, b mergers and c inactions (and arbitrary number of deformations and splittings). The **genus** of the trajectory is defined as

$$b + \frac{a + c}{2}.$$

- ▶ This is inspired by the topological definition of genus: A trajectory traces out a surface over time, and if we start from a single loop and $a = c = 0$, then b gives the number of handles in this surface.

The k^{th} term of our $1/N$ expansion

- ▶ Let ℓ be a loop in \mathbb{Z}^d .
- ▶ For each integer $k \geq 0$, let $\mathcal{X}_k(\ell)$ be the set of all vanishing trajectories of genus $k/2$ that start at the string consisting of the single loop ℓ .
- ▶ Then the k^{th} term in our $1/N$ expansion for the Wilson loop expectation $\langle W_\ell \rangle$ is given by

$$f_k(\ell) = \sum_{X \in \mathcal{X}_k(\ell)} w_\beta(X).$$

Starting point in the proof: The master loop equation

Theorem (C., 2015)

For a string $s = (\ell_1, \dots, \ell_n)$, define

$$\phi(s) := \frac{\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_n} \rangle}{N^n}.$$

Let $|s|$ be the total number of edges in s . Then

$$\begin{aligned}(N-1)|s|\phi(s) &= \sum_{s' \in \mathbb{T}^-(s)} \phi(s') - \sum_{s' \in \mathbb{T}^+(s)} \phi(s') + N \sum_{s' \in \mathbb{S}^-(s)} \phi(s') \\ &\quad - N \sum_{s' \in \mathbb{S}^+(s)} \phi(s') + \frac{1}{N} \sum_{s' \in \mathbb{M}^-(s)} \phi(s') - \frac{1}{N} \sum_{s' \in \mathbb{M}^+(s)} \phi(s') \\ &\quad + N\beta \sum_{s' \in \mathbb{D}^-(s)} \phi(s') - N\beta \sum_{s' \in \mathbb{D}^+(s)} \phi(s').\end{aligned}$$

- ▶ A version of the displayed master loop equation was derived by Makeenko and Migdal in the physics literature in 1979.
- ▶ However, their equation was defined on loops instead of strings, which requires the assumption of an unproved factorization property.
- ▶ The Makeenko–Migdal equation was proved rigorously for 2D Yang–Mills theory by Thierry Lévy in 2011, and later simplified by Driver, Gabriel, Kemp and Hall.
- ▶ The equation displayed in the previous slide is the only rigorous version in general dimensions.
- ▶ My proof is by Stein’s method of exchangeable pairs. Essentially, integration by parts on $SO(N)$.

Key steps in proving the formula for $f_0(\ell)$

- ▶ First, consider the joint limit of $(\phi(s) : s \text{ is a string})$ as $N \rightarrow \infty$.
- ▶ Any limit point must satisfy the limiting master loop equation.
- ▶ Show that if $|\beta|$ is small enough, then there is a unique solution. Thus, limit exists.
- ▶ Assume that the limits are expressible as power series in β (one for each s).
- ▶ Deduce recursive equations for the coefficients in these imagined power series.
- ▶ Define actual power series with coefficients defined through these recursions and show that they converge and satisfy the limiting master loop equation.
- ▶ Thus, by uniqueness, these power series give the limits of $\phi(s)$.
- ▶ Show that the string trajectory formula for $f_0(s)$ can be expressed as a convergent power series in β , and the coefficients satisfy the same recursions as the previous series.

Formula for $f_k(\ell)$

- ▶ Having defined f_0, \dots, f_{k-1} , consider the quantity

$$N^k \left(\phi(s) - f_0(s) - \frac{f_1(s)}{N} - \dots - \frac{f_{k-1}(s)}{N^{k-1}} \right).$$

- ▶ Show that this is uniformly bounded.
- ▶ Derive the limiting master loop equation for this quantity.
- ▶ Carry out similar steps as before to prove the formula for f_k .
More complicated now (three-fold induction instead of two).

Some references

- ▶ Chatterjee, S. (2015). Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large N limit. arXiv:1502.07719. [The formula for the zeroth term in the $1/N$ expansion via string trajectories was derived in this paper.]
- ▶ Chatterjee, S. and Jafarov, J. (2016). The $1/N$ expansion for $SO(N)$ lattice gauge theory at strong coupling. arXiv:1604.04777. [The full $1/N$ expansion was derived here.]
- ▶ Chatterjee, S. (2016). The leading term of the Yang-Mills free energy. arXiv:1602.01222. [Computes the leading term of the free energy of $U(N)$ lattice gauge theory in the continuum limit.]
- ▶ Basu, R. and Ganguly, S. (2016). $SO(N)$ Lattice Gauge Theory, planar and beyond. arXiv:1608.04379. [This paper gives a geometric picture of string trajectories by drawing correspondences to objects such as decorated trees and non-crossing partitions, making it possible to do some explicit calculations using tools from free probability theory and providing hope for further development.]
- ▶ Jafarov, J. (2016). Wilson loop expectations in $SU(N)$ lattice gauge theory. arXiv:1610.03821. [Develops analogous theory for $SU(N)$. More complicated than $SO(N)$.]