Convex polytopes, interacting particles, spin glasses, and finance

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Interacting particles

- Systems of particles governed by joint stochastic differential equations.
- Drift of each particle depends on its relative position with respect to other particles.
- Continuous drifts: classical case (McKean-Vlasov).
- Completely different problem if drifts are prone to abrupt changes.

Example: The Atlas model

- $X_1, \ldots, X_n$ are n diffusive particles, starting from zero.
- At each point of time, the lowest one gets a fixed upward drift.
- Formally,

$$dX_i(t) = \delta_i(t)dt + dB_i(t),$$

  where

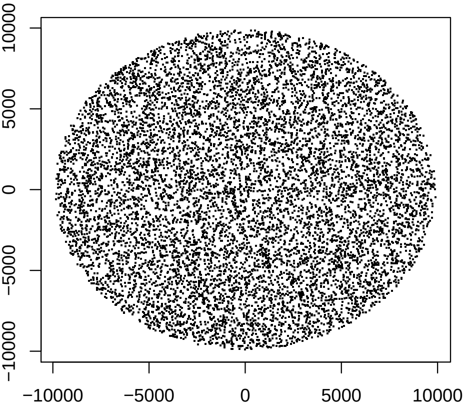
$$\delta_i(t) = \begin{cases} 1 & \text{if } X_i(t) = \min_j X_j(t), \\ 0 & \text{otherwise,} \end{cases}$$

  and $B_1, \ldots, B_n$ are independent Brownian motions.
- Occurs in finance (Banner, Fernholz & Karatzas '05). Uses reflecting Brownian motions and the Harrison-Williams machinery. More later.

Another example

- Again, $X_1, \ldots, X_n$ are n Brownian particles starting from zero.
- At any point of time, the particle that is farthest from zero has a unit drift towards zero.
- Same definition in $\mathbb{R}^d$.
- Question: What is the stationary distribution of this system?
- Cannot use RBM tools in $d \geq 2$.

A snapshot from the stationary law ($n = 10,000$)



Explicit description for finite n

- Generate $U_1, \ldots, U_n$ i.i.d. uniformly from unit ball in $\mathbb{R}^d$.
- Let $V_i = U_i / \max_j \|U_j\|$, $i = 1, \ldots, n$.
- Generate $\Gamma \sim \mathit{Gamma}(nd, 1)$ independently.
- Let
$$Y_i = \frac{1}{2}\Gamma V_i, \ \ i = 1, \ldots, n.$$
Then $(Y_1, \ldots, Y_n)$ follows the stationary distribution of our system.

Solution in $d = 1$

- We have

$$dX_i(t) = \delta_i(t)dt + dB_i(t),$$

where

$$\delta_i(t) = \begin{cases} 1 & \text{if } |X_i(t)| = \max_j |X_j(t)|,\ X_i(t) < 0, \\ -1 & \text{if } |X_i(t)| = \max_j |X_j(t)|,\ X_i(t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Alternatively,

$$d\mathbf{X}(t) = -\nabla k(\mathbf{X}(t))dt + d\mathbf{B}(t),$$

where $k(\mathbf{x}) = \max_i |x_i|$.

Contd...

Proposition

Consider the s.d.e.

$$d\mathbf{X}(t) = -\nabla k(\mathbf{X}(t))dt + d\mathbf{B}(t),$$

where k is any absolutely continuous function. Assume that $\exp(-2k(\mathbf{x}))$ *is integrable and*

$$\int_{\mathbb{R}^n} \|\nabla k(\mathbf{x})\|^2 e^{-2k(\mathbf{x})} d\mathbf{x} < \infty.$$

Then the probability distribution given by the un-normalized density $\exp(-2k(\mathbf{x}))$ *provides a reversible, invariant probability distribution μ for the process $\mathbf{X}(t)$. Under some further conditions (...), the law of $\mathbf{X}(t)$ converges to μ in TV.*

Contd...

- So in our problem, k is the Minkowski norm of the unit cube, and the stationary density is $\propto e^{-2\max_i |x_i|}$.

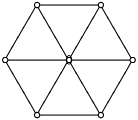
Lemma

If k is the Minkowski norm of a convex set $C \subseteq \mathbb{R}^n$, then picking from $\exp(-2k(\mathbf{x}))$ *is the same as: Picking $\mathbf{U} \sim Unif(C)$ $\rightarrow$ dividing by $2k(\mathbf{U})$ $\rightarrow$ multiplying by independent $Gamma(n, 1)$.*

- Proof is simple, using "polar coordinates" induced by C.
- Picking from $Unif(C)$ is can be handled theoretically if C can be easily triangulated, e.g. if C is a simplicial polytope.

Simplicial polytopes

Each $(n-1)$-dimensional face is a simplex.



- $C = \cup_{j=1}^{d} C_j$; simplices C_j.
- Let $\mathbf{X}_j \sim Unif(C_j)$.
- $\mathbb{P}(\Pi = j) = Vol(C_j)/Vol(C)$.

$\mathbf{X} = \sum_{j=1}^{d} \mathbf{X}_j \, \mathbb{I}(\Pi = j) \sim Unif(C)$.

$$\frac{\Gamma}{2k(\mathbf{X})}\mathbf{X} = \frac{\Gamma}{2k(\mathbf{X}_j)}\mathbf{X}_j, \quad \text{if } \Pi = j.$$

$\dfrac{\Gamma}{2k(\mathbf{X}_j)}\mathbf{X}_j$ - affine transformation of IID Exponentials.

Solving the Atlas model

- Recall:

$$dX_i(t) = \delta_i(t)dt + dB_i(t),$$

where

$$\delta_i(t) = \begin{cases} 1 & \text{if } X_i(t) = \min_j X_j(t), \\ 0 & \text{otherwise.} \end{cases}$$

- Here $k(\mathbf{x}) = -\min_i x_i$.
- This k is not the Minkowski norm of any convex set. Nor is $\exp(-2k(\mathbf{x}))$ integrable.
- The process does not converge to a stationary law.

Atlas model contd...

- Although $k(\mathbf{x}) = -\min_i x_i$ is not a Minkowski norm, it *is* so when restricted to $\sum_i x_i = 0$. On this hyperplane, it is the norm induced by a regular simplex.

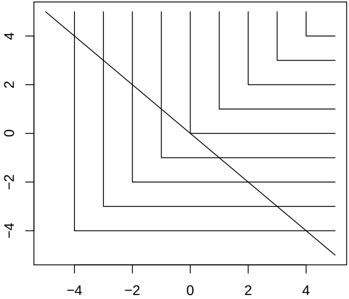


Figure: Level sets of $k(\mathbf{x})$ and their intersection with $\{\mathbf{x} : \sum x_i = 0\}$

Atlas model contd...

- So, we project the process on to $\{\mathbf{x} : \sum x_i = 0\}$ and apply previous result. This gives the stationary law of

$$(X_1(t) - \bar{X}(t), \ldots, X_n(t) - \bar{X}(t)).$$

- Explicit description: Generate i.i.d. exponential r.v.'s with mean $n/2$ and subtract off the mean.
- Obtained by Banner-Fernholz-Karatzas '05, and Pitman (tech. report). Both use Harrison-Williams RBM tools.
- General rank-dependent drifts: Stated as a major open problem in BFK '05. Solved by Pitman (tech. report). Easy solution by the following general result...

A general theorem

Theorem (C. & Pal '06)

Start with $k : \mathbb{R}^n \to \mathbb{R}$.

$$d\mathbf{X}(t) = -\nabla k(\mathbf{X}(t))dt + d\mathbf{B}(t).$$

Suppose there exists a subspace H such that

- $k(x) = k_1(y) + k_2(z), \qquad y = P_H(x), \quad z = x - y.$
- $k_1 \geq 0$, *cont., positively homogenous.*
- $\{x \in H : \; k_1(x) = 0\} = \{0\}.$

Then

- $Y(t) = P_H X(t) \quad \Longrightarrow \quad$ *unique stationary distribution.* *Density* $\propto \exp(-2k_1(y))$. *Can be generated using earlier trick.*
- *Exponentially fast. Reversible when stationary.*

Example: $-\min x_i = -\min(x_i - \bar{x}) - \bar{x}, \;\; H = \{\mathbf{x} : \bar{x} = 0\}.$

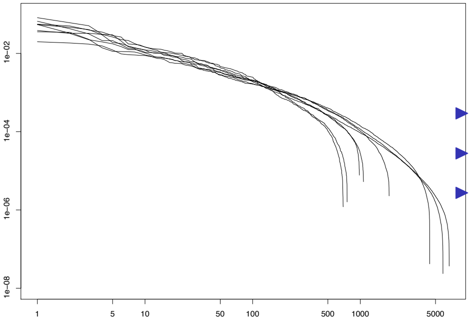
Some finance

- Equity market with n stocks.
- For the ith stock (company), define
  - capital, S_i = number of outstanding shares $\times$ share price
  - market weight,

$$\mu_i = \frac{\text{capital of } i\text{th stock}}{\text{total capital}} = \frac{S_i}{\sum_j S_j}.$$

- Capital distribution curve = log-log plot of μ versus rank.
  - Ordered market wts: $\mu_{(1)} \leq \mu_{(2)} \leq \ldots \leq \mu_{(n)}$.
  - Plot $\log k$ versus $\log \mu_{(n-k+1)}$.
- Economic theory (e.g. Simon '55) predicts that capital distribution curve should be a straight line.
- In reality, the curve is concave and remarkably stable in time.

Real data



- $\log k$ versus $\log \mu_{(n-k+1)}$.
- Dec 31, 1929 - 1999.
- Includes all NYSE, AMEX, and NASDAQ.

Attempted explanations

- Jovanovic ('82), Hopenhayn ('92), Axtell ('99), Hashemi ('00), Kou & Kou ('01).
- However, they all assume that the market converges rapidly to equilibrium, which is never true in reality!
- BFK's Atlas model tries to correct that, but recovers classical curve (straight line).

Stochastic modeling

- Log-capital: $X_i = \log S_i$.
- Black-Scholes: Log-capital of a single company is modeled as a Brownian motion with drift.
- Natural extension for whole market: Log-capitals of all companies modeled as interacting particles on $\mathbb{R}$.
- Empirical observation: Larger stocks have slower upward mobility than smaller stocks.
- One way to model this: The interacting particles (log-capitals) pull each other by a "gravitational force".

We propose: The Gravity Model

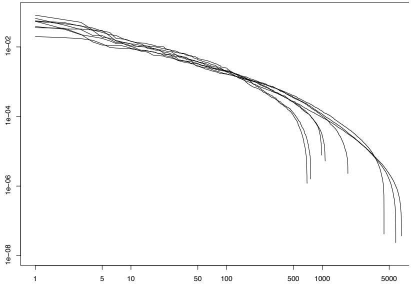
- Let $X_i(t)$, $i = 1, \ldots, n$ be the log-capitals at time t.
- 'Gravitational force' between i and j is proportional to $\mathrm{sign}(X_i - X_j)$.
- Resulting model:

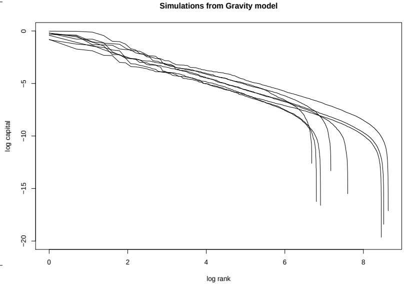
$$dX_i(t) = -\frac{\alpha}{n} \sum_{j=1}^{n} \mathrm{sign}(X_i(t) - X_j(t))dt + dB_i(t), \; i = 1, \ldots, n.$$

- Represents a toy model of flow of capital from larger to smaller stocks. The parameter α determines the strength of the flow.

Capital distribution: Reality vs. Gravity model

Real data (left) vs. simulations from gravity model with $\alpha = 1/2$ (right).





Analysis

- Here

$$k(\mathbf{x}) = \frac{\alpha}{n} \sum_{i,j=1}^{n} |x_i - x_j|.$$

- $k(\mathbf{x})$ is not the Minkowski norm of any convex set in $\mathbb{R}^n$. Also, $\exp(-2k(\mathbf{x}))$ is not integrable.
- However, $k(\mathbf{x})$ is the Minkowski norm of a polytope when restricted to $H = \{\mathbf{x} : \sum x_i = 0\}$.
- The polytope is regular and simplicial. Uniform generation is easy to describe.

Stationary distribution

- $\mathbf{X}(t)$ does not converge in law. However,

$$(X_1(t) - \bar{X}(t), \ldots, X_n(t) - \bar{X}(t))$$

  does converge to an equilibrium measure.
- Suppose $(Y_1, \ldots, Y_n)$ is drawn from this limiting distribution. Let $Y_{(1)} \leq \cdots \leq Y_{(n)}$ denote the Y_i's arranged in increasing order.
- Let $\Delta_i = Y_{(i+1)} - Y_{(i)}$. Then $\Delta_1, \ldots, \Delta_{n-1}$ are independent, and

$$\Delta_i \sim Exp\left(\frac{2\alpha i(n-i)}{n}\right).$$

- Each possible ordering corresponds to one face of the polytope. Simplicial polytope $\Rightarrow$ exponentials.

Phase transition in the gravity model

- Recall: Market weight of ith company is

$$\mu_i(t) = \frac{\text{capital of } i\text{th stock}}{\text{total capital}} = \frac{e^{X_i(t)}}{\sum_j e^{X_j(t)}}.$$

- Market diversity, as defined by Fernholz ('99):

$$\mu_{(n)}(t) := \max_i \mu_i(t) \in [0,1].$$

- Under the gravity model, this has a limiting distribution as $t \to \infty$.

Phase transition contd...

Theorem

Consider the gravity model with strength parameter α:

$$dX_i(t) = -\frac{\alpha}{n}\sum_{j=1}^{n} \mathrm{sign}(X_i(t) - X_j(t))dt + dB_i(t), \; i = 1, \ldots, n.$$

Let $\mu_{(n)}$ denote the diversity in equilibrium. Then as $n \to \infty$,

- *If $\alpha > 1/2$, then $\mu_{(n)} \sim n^{-(2\alpha-1)/2\alpha}$. Diversity exists.*
- *If $\alpha = 1/2$ then $\mu_{(n)} \sim (\log n)^{-1}$. Diversity exists to a lesser extent.*
- *If $\alpha < 1/2$, then $\mu_{(n)} \not\to 0$. No diversity.*

Sketch of proof

- $\mu_{(n)}$ can be written as

$$\frac{1}{1+e^{-\xi_1}+e^{-(\xi_1+\xi_2)}+\cdots+e^{-(\xi_1+\cdots+\xi_{n-1})}},$$

where $\xi_i \sim Exp(2\alpha i(n-i)/n)$.

- When $i \ll n$,

$$\mathbb{E}(\xi_1+\cdots+\xi_i) \sim \frac{\log i}{2\alpha}.$$

- Vague intuition:

$$\mu_{(n)} \sim \frac{1}{1+2^{-1/2\alpha}+3^{-1/2\alpha}+\cdots n^{-1/2\alpha}} \sim \begin{cases} n^{-(2\alpha-1)/2\alpha} & \text{if } \alpha > 1/2, \\ (\log n)^{-1} & \text{if } \alpha = 1/2, \\ \text{const.} & \text{if } \alpha < 1/2. \end{cases}$$

- Rigorous proof via martingales and Poincaré inequalities.

Connection with spin glasses

- Let Σ be the set of possible configurations of a physical system.
- Let $\{Z_\sigma, \sigma \in \Sigma\}$ be a fixed collection of i.i.d. gaussian random variables.
- Derrida's Random Energy Model (REM) assigns a probability measure on Σ by putting mass $\propto \exp(\beta Z_\sigma)$ at each $\sigma \in \Sigma$.
- Exhibits phase transition as β varies. Mathematical reason is the same as for the gravity model phase transition.
- More complex models of spin glasses assume that the Z_σ's have some correlation structure. Similar to more complex versions of the gravity model.

A generalized gravity model

- Gravity model on a graph $G = (V, E)$:

$$dX_i(t) = -\alpha \sum_{j:(i,j)\in E} \mathrm{sign}(X_i(t) - X_j(t))dt + dB_i(t).$$

- As before, $(X_1(t) - \bar{X}(t), \ldots, X_n(t) - \bar{X}(t))$ converges to a stationary law. Polytope is simplicial but not regular.
- Let $(Y_1, \ldots, Y_n)$ be a draw from the stationary distribution. Let $Y_{(1)} \leq \cdots \leq Y_{(n)}$ be the ordered sample.
- Let $\Pi_i =$ the index j such that $Y_{(i)} = Y_j$.
- For each $\pi \in S_n$ and $1 \leq i \leq n-1$, let

$$f_i(\pi) = \#\{(j,k) : j \leq i < k, (\pi_j, \pi_k) \in E\}.$$

- Then $\mathbb{P}(\Pi = \pi) \propto (\prod_{i=1}^{n-1} f_i(\pi))^{-1}$.
- Given $\Pi = \pi$, the increments $Y_{(i+1)} - Y_{(i)}$ are independent and

$$Y_{(i+1)} - Y_{(i)} \sim Exp(2\alpha f_i(\pi)).$$